MAXIMALLY CLUSTERED ELEMENTS AND SCHUBERT VARIETIES

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Abstract. We introduce and study a class of "maximally clustered" elements for simply laced Coxeter groups. Such elements include as a special case the freely braided elements of Green and the author, which in turn constitute a superset of the iji-avoiding elements of Fan. We show that any reduced expression for a maximally clustered element is short-braid equivalent to a "contracted" expression, which can be characterized in terms of certain subwords called "braid clusters". We establish some properties of contracted reduced expressions and apply these to the study of Schubert varieties in the simply laced setting. Specifically, we give a smoothness criterion for Schubert varieties indexed by maximally clustered elements.

1. Introduction

Let W be a simply laced Coxeter group with set S of Coxeter generators. In this paper, we introduce and investigate a notion of "maximally clustered element" for W. Our motivation comes from the articles [5] and [8], where, respectively, the combinatorial properties of short-braid avoiding elements and freely braided elements are applied to the study of Schubert varieties.

Maximally clustered elements are defined in terms of certain triples of root vectors. We explain this briefly as follows. Let $w \in W$. Every reduced expression for w determines a sequence whose terms are the positive roots sent negative by w. If such a "root sequence" has a consecutive subsequence of the form $\alpha, \alpha+\beta, \beta$, then the set containing these three vectors is called a "contractible triple" of w. Some previously studied classes of group elements can be characterized in terms of contractible triples. For example, the iji-avoiding elements of Fan [4] are precisely those elements of W having no contractible triples, and the freely braided elements of Green and the author [6] are the elements of W with pairwise disjoint contractible triples.

We are interested here in elements $w \in W$ with the property that if T and T' are contractible triples of w and $T \cap T' \neq \emptyset$, then the highest roots of T and T' agree. Such a w is said to be "maximally clustered" (the choice of terminology is motivated by Corollary 4.3.3). We show in this paper that every reduced expression for a maximally clustered element is short-braid equivalent to a useful "contracted" expression,

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and for groups W that are finite as well as simply laced, we prove that when a generator is deleted from a contracted reduced expression, the resulting word is reduced unless the deleted generator occurred in the middle of a braid. This enables us to derive a smoothness criterion for Schubert varieties indexed by maximally clustered w. General necessary and sufficient conditions for smoothness of Schubert varieties are known (see, e.g., [10]). However, the criterion given in this paper is rather simple, involving only the support of w and the set of positive roots α sent negative by w such that α is not the sum of two other positive roots sent negative by w.

The paper is organized as follows. In the next section, we establish some terminology and notation related to the general theory of Coxeter groups. In Section 3, after discussing several known properties of contractible triples, we characterize the maximally clustered elements in type A using a pattern avoidance condition (Proposition 3.2.1). We also show that every root sequence for a maximally clustered element is short-braid equivalent to a "contracted" sequence (Corollary 3.3.7). In Section 4, we study contracted reduced expressions, introducing along the way the concepts of "braid cluster" and "braid sequence". In Section 5, we present the deletion result (Theorem 5.2.1) and smoothness criterion (Theorem 5.3.2) mentioned above.

2. Preliminary definitions and notation

Let W be a simply laced Coxeter group with set $S = \{s_i : i \in I\}$ of distinguished generators and Coxeter matrix $(m_{ij})_{i,j \in I}$. The basic facts about Coxeter groups needed for this paper can be found in [1, 9].

2.1. Reduced expressions and root sequences.

Denote by I^* the free monoid on I. We call the elements of I *letters* and those of I^* *words*. The *length* of a word $\mathbf{i} \in I^*$ is the number of factors used to express $\mathbf{i}$ as a product of letters. Let $\phi : I^* \longrightarrow W$ be the surjective morphism of monoid structures determined by the equalities $\phi(i) = s_i$ for all $i \in I$. We say that a word $\mathbf{i} \in I^*$ *represents* its image $\phi(\mathbf{i}) \in W$. If the length of a word $\mathbf{i} \in I^*$ is as small as possible over all words representing $w = \phi(\mathbf{i}) \in W$, then we call $\mathbf{i}$ a *reduced word*, or a *reduced expression* for w. The *length* of $w \in W$, denoted by $\ell(w)$, is the length of any reduced expression for w.

Let V be a real vector space with basis $\{\gamma_i : i \in I\}$ in one-to-one correspondence with I. Let B denote the *Coxeter form* on V associated with $(m_{ij})_{i,j \in I}$. This is the symmetric bilinear form on V satisfying $B(\gamma_i, \gamma_j) = -\cos \frac{\pi}{m_{ij}}$ for all $i, j \in I$. Throughout the paper, we view V as the underlying space of a particular *reflection representation* of W, determined by the equalities $s_i \gamma_j = \gamma_j - 2B(\gamma_j, \gamma_i)\gamma_i$ for all $i, j \in I$. Note that B is preserved by W relative to this representation.

Given $u, v \in V$, we say that u is *orthogonal* to v, and write $u \perp v$, if $B(u, v) = 0$.

Define $\Phi = \{w\gamma_i : i \in I\}$. This is the *root system* of W. The elements of Φ, i.e., the *roots*, all have unit length relative to B. The basis vectors γ_i are called *simple* roots. Let Φ^+ (resp., Φ^-) denote the set of all roots expressible as a linear combination of simple roots with nonnegative (resp., nonpositive) coefficients. This is the set of *positive* (resp., *negative*) roots. We have $\Phi = \Phi^+ \cup \Phi^-$ (disjoint).

Let $w \in W$. Denote by $\Phi(w)$ the set of all $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$. The cardinality of $\Phi(w)$ is $\ell(w)$. Given any reduced expression $i_1 \cdots i_n$ for w, we have $\Phi(w) = \{r_1, \dots, r_n\}$, where $r_1 = \gamma_{i_n}$ and $r_q = s_{i_n} \cdots s_{i_{n-q+2}}(\gamma_{i_{n-q+1}})$ for all $q \in \{2, \dots, n\}$. The sequence $\overline{r} = (r_1, \dots, r_n)$ is called the *root sequence* of $i_1 \cdots i_n$, or a root sequence *for* w. Note that for any $1 \leq q \leq n$, the initial segment $(r_1, \dots, r_q)$ of $\overline{r}$ is the root sequence of $i_{n-q+1} \cdots i_n$. Observe also that every reduced word is uniquely determined by its root sequence, so that the map $i_1 \cdots i_n \mapsto \overline{r}$ is a bijection from the set of all reduced expressions for w to the set of all root sequences for w.

2.2. **Braid moves.** For any letters $i, j \in I$ and any positive integer q, we define $(i, j)_q$ to be the length q word $iji \cdots \in I^*$.

Let $\mathbf{i}, \mathbf{j} \in I^*$ and let $i, j, k \in I$ with $i \neq j$. Denote the length of $\mathbf{j}$ by n. We call the substitution $\mathbf{i}(i, j)_{m_{ij}}\mathbf{j} \to \mathbf{i}(j, i)_{m_{ji}}\mathbf{j}$ a *braid move*, qualifying it *short* or *long* according as $m_{ij} = 2$ or 3, respectively. Braid moves can be described in terms of root sequences [6, Prop. 3.1.1], as follows. Suppose that the word $\mathbf{i}ij\mathbf{j}$ is reduced, and let $\overline{r} = (r_q)$ be its root sequence. Then $m_{ij} = 2$ if and only if $r_{n+1} \perp r_{n+2}$, and when these conditions hold, $\mathbf{i}ji\mathbf{j}$ is reduced and its root sequence $\overline{r}'$ can be obtained from $\overline{r}$ by swapping r_{n+1} and r_{n+2}. We call the passage from $\overline{r}$ to $\overline{r}'$ (assuming $r_{n+1} \perp r_{n+2}$) a *short braid move.* Suppose now that the word $\mathbf{i}ijk\mathbf{j}$ is reduced, and let $\overline{r} = (r_q)$ be its root sequence. Then $i = k$ if and only if $r_{n+1} + r_{n+3} = r_{n+2}$, and when these conditions hold, $\mathbf{i}jij\mathbf{j}$ is reduced and its root sequence $\overline{r}'$ is obtainable from $\overline{r}$ by swapping r_{n+1} and r_{n+3}. Here, we call the passage from $\overline{r}$ to $\overline{r}'$ (assuming $r_{n+1} + r_{n+3} = r_{n+2}$) a *long braid move.* Our definitions are such that the bijection at the end of Section 2.1 is compatible with both long and short braid moves.

Let $w \in W$. We shall find useful a theorem of Matsumoto [13] and Tits [16], which states that every reduced expression for w can be obtained from any other through a sequence of braid moves. A similar statement holds for root sequences, on account of the aforementioned bijection.

We say that two words in I^* are *short-braid equivalent* if one can be obtained from the other by a sequence of short braid moves. There is a corresponding notion for root sequences.

3. Contracted root sequences

We begin this section by introducing contractible triples and maximally clustered elements, and then we give an alternate description of maximally clustered elements in type A using a pattern avoidance condition. The rest of the section will be devoted to showing that every root sequence for a maximally clustered element is short-braid equivalent to a "contracted" sequence.

3.1. **Contractible triples.** The *height* of any $\alpha \in \Phi$ is the sum of the coefficients used to express α as a linear combination of simple roots. The following definition was first presented in [6] (except for the statistic $\widetilde{N}$, which is new).

Definition 3.1.1. Let $w \in W$. We call any subset of $\Phi(w)$ of the form $\{\alpha, \beta, \alpha + \beta\}$ an *inversion triple* of w. If T is an inversion triple of w such that there is a root sequence for w in which the elements of T appear consecutively (in some order), then we say that T is *contractible*. We denote by $N(w)$ the number of contractible inversion triples of w, and by $\widetilde{N}(w)$ the number of roots α such that α is the highest root of at least one contractible inversion triple of w.

For brevity, we usually write "contractible triple" instead of "contractible inversion triple". Observe that $\widetilde{N}(w) \leq N(w)$ for all $w \in W$. Another description of $\widetilde{N}(w)$ is given for certain w in Remark 5.1.3.

Remark 3.1.2. In the type A setting, every inversion triple is contractible, but this does not hold in general [6, Example 2.2.3, Prop. 5.1.1].

Remark 3.1.3. Suppose that $(\dots, \alpha, \gamma, \beta, \dots)$ is a root sequence for some $w \in W$, and the following long braid move can be applied: $(\dots, \alpha, \gamma, \beta, \dots) \to (\dots, \beta, \gamma, \alpha, \dots)$. Then $\{\alpha, \beta, \gamma\}$ is a contractible triple of w with highest root γ.

Remark 3.1.4. Let $w \in W$. Suppose that $\alpha, \beta \in \Phi^+$ are such that $\alpha + \beta \in \Phi$. By the linearity of $w|_V$, the following implications hold: $\alpha, \beta \in \Phi(w) \Rightarrow \alpha + \beta \in \Phi(w)$ and $\alpha, \beta \notin \Phi(w) \Rightarrow \alpha + \beta \notin \Phi(w)$.

Now suppose that $\{\alpha, \beta, \alpha + \beta\}$ is an inversion triple of w. In every root sequence for w, the root $\alpha + \beta$ must appear between α and β, by the previous paragraph and the fact that every initial segment of any root sequence is again a root sequence.

Remark 3.1.5. If $\alpha, \beta, \alpha + \beta \in \Phi$, then $\alpha \not\perp (\alpha + \beta)$: otherwise, $1 = B(\beta, \beta) = B((\alpha+\beta) - \alpha, (\alpha+\beta) - \alpha) = B(\alpha+\beta, \alpha+\beta) - 2B(\alpha+\beta, \alpha) + B(\alpha, \alpha) = 1 - 0 + 1 = 2$.

We also have $2\alpha + \beta \notin \Phi$. To see this, assume the contrary. Then $1 = B(2\alpha + \beta, 2\alpha+\beta) = B(2\alpha, 2\alpha) + 2B(2\alpha, \beta) + B(\beta, \beta) = 4 + 4B(\alpha, \beta) + 1$, so that $B(\alpha, \beta) = -1$. Hence, $B(\alpha, \alpha + \beta) = B(\alpha, \alpha) + B(\alpha, \beta) = 0$, contradicting the previous paragraph.

By symmetry, the above holds with α and β interchanged. It follows that if T and T' are inversion triples of the same element of W and $\#(T \cap T') > 1$, then $T = T'$.

The situation where the contractible triples of an element w are pairwise disjoint has been considered in [6, 7, 8] (and in the special case where the Coxeter group is of type A, in [12]; see also [15, 17]). Such w are said to be *freely braided*. Observe that $\widetilde{N}(w) = N(w)$ for any freely braided w. An example of a non-freely braided element w with $\widetilde{N}(w) = N(w)$ is given in the last paragraph of Section 3.2.

Given $w \in W$, we have $N(w) = 0$ if and only if no reduced expression for w is of the form $\mathbf{i}iji\mathbf{j}$ where $\mathbf{i}, \mathbf{j} \in I^*$ and $i, j \in I$ (this fact, which was recorded in [7, Prop. 1.2.2], follows easily from Section 2.2). Such "iji-avoiding" elements w were introduced by Fan [4], and have the property that every reduced expression for w is short-braid equivalent to any other. In the setting of arbitrary (including non-simply laced) Coxeter groups, Stembridge [14] has used this last property to define a notion of "fully commutative" element. Note that every iji-avoiding element is freely braided.

Definition 3.1.6. Let $w \in W$. Suppose that for any pair of intersecting contractible triples T and T' of w, the highest roots in T and T' agree. Then we say that w is *maximally clustered*.

The choice of terminology is explained by Corollary 4.3.3.

Observe that every freely braided group element is maximally clustered, and that any maximally clustered w with $\widetilde{N}(w) = N(w)$ is freely braided. An example of a non-freely braided, maximally clustered element is given at the end of Section 3.2.

3.2. **Type A.** Assume that W is of type A. Identify W with the group $\mathcal{S}_n$ of permutations of $\{1, \ldots, n\}$, and S with the set of simple transpositions $s_1 = (1,2), \ldots, s_{n-1} = (n-1, n)$. The set I of letters is thus identified with $\{1, \ldots, n-1\}$. In $\mathbb{R}^n$, denote the standard basis vectors by $\varepsilon_1, \ldots, \varepsilon_n$, and the standard inner product by $\langle \cdot, \cdot \rangle$. Let $\gamma_q = \varepsilon_q - \varepsilon_{q+1}$, for $1 \leq q \leq n-1$. Take V to be the subspace of $\mathbb{R}^n$ spanned by $\gamma_1, \ldots, \gamma_{n-1}$. We have $B(\gamma_i, \gamma_j) = \frac{1}{2}\langle \gamma_i, \gamma_j \rangle$ for all $i, j \in I$. The linear action $\mathcal{S}_n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying $(\pi, \varepsilon_q) \mapsto \varepsilon_{\pi(q)}$ for all $\pi \in \mathcal{S}_n$ and $1 \leq q \leq n$ restricts to an action $\mathcal{S}_n \times V \longrightarrow V$, and the latter corresponds to the reflection representation described in Section 2.1. The simple roots are $\gamma_1, \ldots, \gamma_{n-1}$, and we have $\Phi = \{\varepsilon_p - \varepsilon_q : 1 \leq p \neq q \leq n\}$ and $\Phi^+ = \{\varepsilon_p - \varepsilon_q : 1 \leq p < q \leq n\}$.

Let $w \in \mathcal{S}_n$ and let p, q be integers with $1 \leq p < q \leq n$. We have $\varepsilon_p - \varepsilon_q \in \Phi(w) \Leftrightarrow w(p) > w(q)$. Hence, in view of Remark 3.1.2, a triple $\{\varepsilon_p - \varepsilon_q, \varepsilon_q - \varepsilon_r, \varepsilon_p - \varepsilon_r\}$ of elements of Φ^+ is a contractible triple of w if and only if $w(p) > w(q) > w(r)$.

Let $w \in \mathcal{S}_n$ and $v \in \mathcal{S}_k$ with $k \leq n$. Suppose that there exist indices $1 \leq q_1 < \cdots < q_k \leq n$ such that the numbers $w(q_1), w(q_2), \ldots, w(q_k)$ are in the same relative order as $v(1), v(2), \ldots, v(k)$. Then we say that w *contains the pattern* $v(1)v(2)\cdots v(k)$. We also say that $w(q_1)w(q_2)\cdots w(q_k)$ *is an occurrence of* $v(1)v(2)\cdots v(k)$ in w. If such indices do not exist, then we say that w *avoids* $v(1)v(2)\cdots v(k)$.

Proposition 3.2.1. *A permutation $w \in \mathcal{S}_n$ is maximally clustered if and only if it avoids the patterns* 4321, 3421 *and* 4312.

Proof. Assume that w is not maximally clustered, and let $T = \{\varepsilon_p - \varepsilon_q, \varepsilon_q - \varepsilon_r, \varepsilon_p - \varepsilon_r\}$ be a contractible triple of w such that one of the two shorter roots in T belongs to another contractible triple T' of w. Note that $w(p) > w(q) > w(r)$.

Suppose that $\varepsilon_p - \varepsilon_q \in T \cap T'$. If $\varepsilon_p - \varepsilon_q$ is the highest root of T', then there exists $p < l < q$ such that $w(p) > w(l) > w(q)$. The string $w(p)w(l)w(q)w(r)$ is then an occurrence of 4321 in w. If $\varepsilon_p - \varepsilon_q$ is not the highest root of T', then there are three cases to consider. If there exists $1 \leq l < p$ with $\varepsilon_l - \varepsilon_p \in T'$, then $w(l)w(p)w(q)w(r)$

is an occurrence of 4321 in w. If there exists $q < l < r$ with $\varepsilon_q - \varepsilon_l \in T'$, then $w(p)w(q)w(l)w(r)$ is an occurrence of either 4321 or 4312. Finally, if there exists $r < l \leq n$ with $\varepsilon_q - \varepsilon_l \in T'$, then $w(p)w(q)w(r)w(l)$ is an occurrence of 4321 or 4312.

By a similar analysis, if $\varepsilon_q - \varepsilon_r \in T'$, then w contains 4321 or 3421.

We omit the straightforward verification that w is not maximally clustered if it contains at least one of the three patterns in the statement. □

A permutation is freely braided (as defined in Section 3.1) if and only if it avoids the patterns 4321, 3421, 4231 and 4312 [6, Sect. 5.1]. Thus, a maximally clustered permutation is non-freely braided precisely when it contains 4231. Note that the permutation 3421 is not maximally clustered, and hence not freely braided, yet $\widetilde{N}(3421) = 2 = N(3421)$ (the same holds for 4312).

3.3. **Contracted root sequences.** Assume that W is of arbitrary simply laced type. Our current aim is to show that every root sequence for a maximally clustered element is short-braid equivalent to a sequence in which the contractible triples of the element have been gathered together into various non-overlapping segments, the triples in each segment sharing the same highest root.

Lemma 3.3.1. *Let $w \in W$ be maximally clustered. Suppose that $T = \{\alpha, \beta, \alpha+\beta\}$ is a contractible triple of w, and that $\gamma \in \Phi(w) \setminus T$ occurs between α and β in some root sequence $\overline{r}$ for w. Then $\gamma \perp \alpha$ or $\gamma \perp \beta$. If γ occurs between α and $\alpha+\beta$ in $\overline{r}$, and γ does not lie in any contractible triple of w containing $\alpha+\beta$, then $\gamma \perp \alpha$.*

Proof. Consider a sequence of braid moves carrying $\overline{r}$ to a root sequence in which the elements of T occur consecutively (such a sequence exists by the contractibility of T and the theorem of Matsumoto and Tits mentioned in Section 2.2). There is a braid move in the sequence reversing the relative order of α and γ, or of γ and β. However, such a move cannot be long. To see this, assume otherwise. Then, by Remark 3.1.3, there is a contractible triple T' of w such that $\{\alpha, \gamma\} \subseteq T'$ or $\{\beta, \gamma\} \subseteq T'$. Hence, $\alpha \in T \cap T'$ or $\beta \in T \cap T'$. Since $T \neq T'$ ($\gamma \in T' \setminus T$), we cannot have $\alpha+\beta \in T'$ (by the last paragraph of Remark 3.1.5), and so w is not maximally clustered, a contradiction. Thus, there is a short braid move in our sequence of moves swapping either γ and α, or γ and β, and from this we obtain $\gamma \perp \alpha$ or $\gamma \perp \beta$.

Assume now that (i) γ occurs between α and $\alpha+\beta$ in $\overline{r}$, that (ii) γ does not belong to any contractible triple of w containing $\alpha+\beta$, and, toward a contradiction, that (iii) $\gamma \not\perp \alpha$. We continue to work with the sequence of braid moves from the previous paragraph. By (iii) together with the previous paragraph, no braid move in our sequence reverses the relative order of γ and α, whereas γ and β are swapped by some short braid move. Together with (i) and Remark 3.1.4, this implies the existence of an earlier move reversing the relative order of γ and $\alpha+\beta$. Using (ii) and Remark 3.1.3, we deduce that γ and $\alpha+\beta$ are swapped by a short braid move, so that $\gamma \perp (\alpha+\beta)$. Combining this with (iii) gives $\gamma \not\perp \beta$, a contradiction. □

Lemma 3.3.2. *Let $w \in W$ be maximally clustered. Suppose that $\{\alpha, \beta, \alpha+\beta\}$ and $\{\alpha', \beta', \alpha'+\beta'\}$ are distinct contractible triples of w with $\alpha+\beta = \alpha'+\beta'$. Then α' is orthogonal to precisely one of α, β. If $\alpha' \perp \alpha$, then $\beta' \perp \beta$ and $\beta' \not\perp \alpha$.*

Proof. First note that α' cannot be orthogonal to both α and β, since otherwise α' would be orthogonal to $\alpha+\beta = \alpha'+\beta'$, contradicting Remark 3.1.5. By the contractibility of $\{\alpha', \beta', \alpha'+\beta'\}$ and Remark 3.1.4, there is a root sequence for w in which α' occurs between α and β. The first assertion of the lemma now follows from Lemma 3.3.1.

By symmetry, β' is orthogonal to precisely one of α, β. If $\alpha' \perp \alpha$, then $\beta' \not\perp \alpha$, since otherwise we would have $\alpha \perp (\alpha'+\beta' = \alpha+\beta)$. □

Lemma 3.3.3. *Let $w \in W$ be maximally clustered. Suppose that $\{\alpha, \beta, \alpha+\beta\}$ and $\{\alpha', \beta', \alpha'+\beta'\}$ are distinct contractible triples of w with $\alpha+\beta = \alpha'+\beta'$. Suppose also that $\overline{r}$ is a root sequence for w of the form*

$$\overline{r} = (\ldots, \alpha', \ldots, \alpha, \ldots, \alpha+\beta = \alpha'+\beta', \ldots, \beta', \ldots, \beta, \ldots).$$

Then $\alpha' \perp \alpha$ and $\beta' \perp \beta$.

Proof. By Lemma 3.3.2, either the conclusion holds, or $\alpha' \perp \beta$, $\alpha' \not\perp \alpha$, $\beta' \perp \alpha$ and $\beta' \not\perp \beta$. Assume the latter.

Consider a sequence of braid moves carrying $\overline{r}$ to a root sequence in which the elements of $\{\alpha', \beta', \alpha'+\beta'\}$ occur consecutively. By the previous paragraph and the fact that w is maximally clustered, no braid move in the sequence reverses the relative order of α and α', whereas α and β' are swapped by some short braid move. Hence, the relative order of α and $\alpha+\beta$ is reversed at some stage. This implies by Remark 3.1.5 the existence of an earlier braid move reversing the relative order of β and β'. Since w is maximally clustered, we have $\beta' \perp \beta$, a contradiction. □

Definition 3.3.4. Let $w \in W$ be maximally clustered. Let C be a collection of pairwise-intersecting contractible triples of w. Suppose that C is nonempty and not properly contained in another set of pairwise-intersecting contractible triples of w. Then we call C a *maximal set of triples* for w.

Let $w \in W$ be maximally clustered, and suppose that C and C' are distinct maximal sets of triples for w. Then $C \cap C' = \emptyset$. In fact, for any $T \in C$ and $T' \in C'$, we have $T \cap T' = \emptyset$. If $C_1, \ldots, C_n$ is a complete, non-repeating list of the maximal sets of triples for w, then $N(w) = \# \bigcup_{q=1}^{n} C_q = \sum_{q=1}^{n} \# C_q$. Note that $n = \widetilde{N}(w)$.

Proposition 3.3.5. *Let $w \in W$ be maximally clustered, let $\overline{r}$ be a root sequence for w, and suppose that $C = \{T_1, \ldots, T_n\}$ is a maximal set of triples for w. For each $1 \leq q \leq n$, write $T_q = \{\alpha_q, \beta_q, \gamma\}$, where $\gamma = \alpha_1 + \beta_1 = \cdots = \alpha_n + \beta_n$. Assume without loss of generality that the labelling is such that*

$$\overline{r} = (\ldots, \alpha_1, \ldots, \alpha_2, \ldots, \alpha_n, \ldots, \gamma, \ldots, \beta_{\pi(1)}, \ldots, \beta_{\pi(2)}, \ldots, \beta_{\pi(n)}, \ldots),$$

where $\pi \in \mathcal{S}_n$. Then $\overline{r}$ is short-braid equivalent to a root sequence of the form

$$(\ldots, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n, \gamma, \beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1, \ldots).$$

It is also short-braid equivalent to a sequence of the form

$$(\ldots, \alpha_{\pi(n)}, \alpha_{\pi(n-1)}, \ldots, \alpha_{\pi(2)}, \alpha_{\pi(1)}, \gamma, \beta_{\pi(1)}, \beta_{\pi(2)}, \ldots, \beta_{\pi(n-1)}, \beta_{\pi(n)}, \ldots).$$

Proof. By Lemma 3.3.1, there is a sequence of short braid moves, each swapping α_n with the entry directly to its right, carrying $\overline{r}$ to a sequence $\overline{r}'$ in which α_n sits directly to the left of γ. By Lemmas 3.3.1 and 3.3.3, there is a sequence of short braid moves, each swapping β_n with the entry directly to its left, carrying $\overline{r}'$ to a sequence in which the roots $\alpha_n, \gamma, \beta_n$ appear consecutively. Continuing, we obtain a root sequence as in the first part of the conclusion.

A sequence as in the second part of the conclusion can be obtained from $\overline{r}$ in a similar way. ☐

Definition 3.3.6. Let $w \in W$ be maximally clustered, and let $\overline{r}$ be a root sequence for w. Suppose that C is a maximal set of triples for w, and that

$$\overline{r} = (\ldots, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n, \gamma, \beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1, \ldots),$$

where $\{\{\alpha_1, \beta_1, \gamma\}, \ldots, \{\alpha_n, \beta_n, \gamma\}\} = C$. Then we say that $\overline{r}$ is *contracted for* C. We say that a root sequence for w is *contracted* if it is contracted for every maximal set of triples for w.

Corollary 3.3.7. *Let $w \in W$ be maximally clustered. Then every root sequence for w is short-braid equivalent to a contracted root sequence.*

Proof. Let $\overline{r}$ be a root sequence for w, and suppose that $C_1, \ldots, C_n$ is a complete, non-repeating list of the maximal sets of triples for w. Apply (the proof of) Proposition 3.3.5 repeatedly, once for each C_q, starting with the sequence $\overline{r}$. The result is a sequence that is short-braid equivalent to $\overline{r}$ and, by the paragraph following Definition 3.3.4, contracted for all C_q. ☐

The conditions in the following proposition will reappear in Section 4.2.

Proposition 3.3.8. *Suppose that $w \in W$ is maximally clustered and has a root sequence of the form $(\ldots, \alpha_1, \ldots, \alpha_2, \ldots, \alpha_3, \ldots, \gamma, \ldots)$, where, for each $1 \leq q \leq 3$, $\{\alpha_q, \gamma, \gamma - \alpha_q\}$ is a contractible triple of w. Then the following statements hold:*

(i) *If $\alpha_2 \perp \alpha_3$, then $\alpha_1 \perp \alpha_2$ or $\alpha_1 \perp \alpha_3$.*
(ii) *If $\alpha_1 \perp \alpha_3$, then $\alpha_1 \perp \alpha_2$ or $\alpha_2 \perp \alpha_3$.*

Proof. For (i), assume toward a contradiction that $\alpha_2 \perp \alpha_3$, $\alpha_1 \not\perp \alpha_2$ and $\alpha_1 \not\perp \alpha_3$. Then $\alpha_2 \not\perp (\gamma - \alpha_3)$ by Lemma 3.3.2. Since w is maximally clustered, in every root sequence for w, the root α_1 is to the left of both α_2 and α_3, and α_2 is to the left of $\gamma - \alpha_3$. Since $\{\alpha_1, \gamma, \gamma - \alpha_1\}$ is contractible, there is a root sequence $\overline{r}$ for w of the form $(\ldots, \alpha_1, \gamma, \gamma - \alpha_1, \ldots)$. In this sequence, both α_3 and $\gamma - \alpha_3$ appear to the right of γ, which contradicts Remark 3.1.4. Thus, (i) holds.

Assertion (ii) can be proved in a similar way. □

4. Braid clusters and braid sequences

In this section, we give a detailed description of the reduced expressions that correspond to contracted root sequences. Assume throughout that W is of arbitrary simply laced type. Note that some of the results will be stated and proved for all (not just maximally clustered) $w \in W$.

4.1. **Braid clusters.** A *subword* of a given word $i_1 i_2 \cdots i_n$ (each $i_l \in I$) is any word of the form $i_p i_{p+1} \cdots i_q$ where $1 \leq p \leq q \leq n$.

Definition 4.1.1. Suppose that $\mathbf{i} \in I^*$ is of the form $\mathbf{i} = i_1 i_2 \cdots i_n i_{n+1} i_n \cdots i_2 i_1$ ($n \geq 1$), where $i_1, i_2, \ldots, i_{n+1} \in I$ are distinct and where, for each $1 \leq q \leq n$, there is a unique $q < r \leq n+1$ such that $m_{i_q i_r} = 3$. We call $\mathbf{i}$ a *braid cluster.*

If $i_1 i_2 \cdots i_n i_{n+1} i_n \cdots i_2 i_1$ is a braid cluster, then so is $i_q i_{q+1} \cdots i_n i_{n+1} i_n \cdots i_{q+1} i_q$ for all $1 \leq q \leq n$. The following definition presents two types of moves. The reader will notice that each, when applied to a braid cluster, results in another braid cluster.

Definition 4.1.2. Let $\mathbf{i} \in I^*$ be a word of the form $\mathbf{i} = i_1 i_2 \cdots i_n i_{n+1} i_n \cdots i_2 i_1$, where $i_1, \ldots, i_{n+1} \in I$ (we are not requiring that $\mathbf{i}$ be a braid cluster here).

- (i) A *type 1 move* on $\mathbf{i}$ consists of two short braid moves, one exchanging the letters in positions q and $q+1$ (assuming $q < n$ and $m_{i_q i_{q+1}} = 2$), and the other swapping the letters in positions $2n-q+1$ and $2n-q+2$.
- (ii) A *type 2 move* is a long braid move, applied to the middle three letters of $\mathbf{i}$ (assuming $m_{i_n i_{n+1}} = 3$).

In what follows, we say that $i, j \in I$ are *m-commuting* if $m_{ij} < 3$.

Lemma 4.1.3. *Suppose that* $\mathbf{i} = i_1 i_2 \cdots i_n i_{n+1} i_n \cdots i_2 i_1 \in I^*$ *is a braid cluster. Then the following statements hold:*

- (i) $\mathbf{i}$ *is reduced.*
- (ii) *For each* $1 \leq q \leq n$*, there is a sequence of type 1 and 2 moves carrying* $i_q i_{q+1} \cdots i_n i_{n+1} i_n \cdots i_{q+1} i_q$ *to a braid cluster in which the middle letter is* i_q.

Proof. Observe, since $\mathbf{i}$ is a braid cluster, when $\phi(i_2 \cdots i_n i_{n+1} i_n \cdots i_2)(\gamma_{i_1})$ is written as a linear combination of simple roots, $\gamma_{i_{n+1}}$ appears with coefficient 1. Hence, by a basic result of Coxeter group theory [9, Thm. 5.4], if $i_2 \cdots i_n i_{n+1} i_n \cdots i_2$ is reduced, then so is $\mathbf{i}' = i_2 \cdots i_n i_{n+1} i_n \cdots i_2 i_1$. By similar reasoning, $\mathbf{i} = i_1 \mathbf{i}'$ is reduced if $\mathbf{i}'$ is.

Statement (i) now follows by induction on $n \geq 1$. In fact, the above establishes the base case $\mathbf{i} = i_1 i_2 i_1$, since i_2 is reduced, as well as the induction step.

We also prove (ii) by induction on n. The assertion holds for $n = 1$ (one applies a type 2 move), and also whenever $1 < q \leq n$ by the inductive hypothesis. Suppose that $1 = q < n$. If $m_{i_1 i_2} = 2$, then apply a type 1 move to the outermost letters of $\mathbf{i}$, followed by the inductive hypothesis. If instead $m_{i_1 i_2} = 3$, then use the inductive hypothesis to transform $i_2 \cdots i_n i_{n+1} i_n \cdots i_2$ into a cluster $\mathbf{i}''$ in which i_2 is the middle letter. The first two letters of $i_1 \mathbf{i}'' i_1$ now m-commute, so we can proceed as in the case $m_{i_1 i_2} = 2$. □

Remark 4.1.4. One could define "braid cluster" more generally to mean any word $\mathbf{j}$ that is short-braid equivalent to a word $\mathbf{i}$ as in Definition 4.1.1. For such a $\mathbf{j}$, any reduced expression for $\phi(\mathbf{j})$ would be a braid cluster. One might then call an element $w \in W$ a braid cluster if some (every) reduced expression for w is a braid cluster (w would be a reflection). In this paper, however, we understand "braid cluster" always to mean a word $\mathbf{i}$ as in Definition 4.1.1.

4.2. **Braid sequences.**

For us, a *subsequence* of a root sequence $(r_1, r_2, \ldots, r_n)$ is any sequence of the form $(r_p, r_{p+1}, \ldots, r_q)$ where $1 \le p \le q \le n$.

The conditions in the following definition are reminiscent of Lemma 3.3.2 and Proposition 3.3.8, although we are not requiring w to be maximally clustered here.

Definition 4.2.1. Let $w \in W$, and let $\overline{r}$ be a root sequence for w. Suppose that $\overline{r}$ has a subsequence of the form $(\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$, where $n \ge 1$ and

(i) $\alpha_q + \beta_q = \gamma$ for all $1 \le q \le n$.
(ii) For any $1 \le p < q < r \le n$, if $\alpha_q \perp \alpha_r$, then $\alpha_p \perp \alpha_q$ or $\alpha_p \perp \alpha_r$.
(iii) For any $1 \le p < q < r \le n$, if $\alpha_p \perp \alpha_r$, then $\alpha_p \perp \alpha_q$ or $\alpha_q \perp \alpha_r$.
(iv) For any $1 \le p \ne q \le n$, each of α_p, β_p is orthogonal to precisely one of α_q, β_q.

Then we say that $(\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$ is a *braid sequence* in $\overline{r}$, or a braid sequence *for* w.

Observe that if $(\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$ is a braid sequence for $w \in W$, then so is $(\alpha_q, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_q)$ for all $1 \le q \le n$. The moves described below are meant to correspond to those described in Definition 4.1.2.

Definition 4.2.2. Suppose that $\overline{a} = (\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$ is a subsequence of some root sequence ($\overline{a}$ is not necessarily a braid sequence). We define two types of moves.

(i) A *type $\overline{1}$ move* on $\overline{a}$ consists of two short braid moves, one interchanging α_q and α_{q+1}, and the other interchanging β_{q+1} and β_q (for $1 \le q < n$ and assuming $\alpha_q \perp \alpha_{q+1}$ and $\beta_q \perp \beta_{q+1}$).
(ii) A *type $\overline{2}$ move* on $\overline{a}$ is a long braid move, swapping α_n and β_n (assuming $\alpha_n + \beta_n = \gamma$).

The following two lemmas are needed for our work in Section 4.3, where we discuss the correspondence between braid clusters and braid sequences.

Lemma 4.2.3. *Suppose that $\overline{a} = (\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$ is a braid sequence for some $w \in W$. Then the application of any type $\overline{1}$ or $\overline{2}$ move to $\overline{a}$ results in another braid sequence for w.*

Proof. Suppose that a type $\overline{1}$ or $\overline{2}$ move is applied to $\overline{a}$, and denote the resulting sequence by $\overline{a}' = (\alpha_1', \dots, \alpha_n', \gamma, \beta_n', \dots, \beta_1')$. One sees immediately that conditions (i) and (iv) of Definition 4.2.1 are satisfied by $\overline{a}'$.

Assume for a contradiction that condition (ii) of the definition fails, and let $1 \le p < q < r \le n$ be such that $\alpha_q' \perp \alpha_r'$, $\alpha_p' \not\perp \alpha_q'$ and $\alpha_p' \not\perp \alpha_r'$. If $\overline{a}'$ was obtained from $\overline{a}$ by a type $\overline{1}$ move, then this move must have involved two of the roots $\alpha_p', \alpha_q', \alpha_r'$ (otherwise, the relative order of $\alpha_p', \alpha_q', \alpha_r', \gamma$ would be the same in $\overline{a}$ as in $\overline{a}'$, contradicting the fact that $\overline{a}$ is a braid sequence). By the above orthogonality relations, the roots α_q' and α_r' must have been interchanged. Thus, $\alpha_p' = \alpha_p$, $\alpha_q' = \alpha_r$ and $\alpha_r' = \alpha_q$, and so condition (ii) of Definition 4.2.1 fails for $\overline{a}$ relative to p, q, r, a contradiction.

If instead $\overline{a}'$ was obtained from $\overline{a}$ by a type $\overline{2}$ move, then α_r' and β_r' must have been swapped (or else the parenthetical remark from the previous paragraph would apply here). This means that $\alpha_p' = \alpha_p$, $\alpha_q' = \alpha_q$ and $\alpha_r' = \beta_r$. Now, $\overline{a}$, being a braid sequence, satisfies condition (iv) of Definition 4.2.1. Hence, the orthogonality relations from the previous paragraph give us $\alpha_q \not\perp \alpha_r$, $\alpha_p \not\perp \alpha_q$ and $\alpha_p \perp \alpha_r$, so that condition (iii) of Definition 4.2.1 fails for $\overline{a}$ relative to p, q, r, an impossibility. Thus, condition (ii) of Definition 4.2.1 holds for $\overline{a}'$.

To see that condition (iii) holds for $\overline{a}'$, assume otherwise, and let $1 \le p < q < r \le n$ be such that $\alpha_p' \perp \alpha_r'$, $\alpha_p' \not\perp \alpha_q'$ and $\alpha_q' \not\perp \alpha_r'$. Note that $\overline{a}'$ could not have been obtained from $\overline{a}$ by a type $\overline{1}$ move (because such a move could not involve fewer than two of the roots $\alpha_p', \alpha_q', \alpha_r'$ by the parenthetical remark two paragraphs above, and it could not involve two of the roots by the above orthogonality relations). Thus, $\overline{a}'$ was obtained from $\overline{a}$ by a type $\overline{2}$ move, so that $\alpha_p' = \alpha_p$, $\alpha_q' = \alpha_q$ and $\alpha_r' = \beta_r$. Once again, we combine the fact that $\overline{a}$ satisfies condition (iv) of Definition 4.2.1 with our orthogonality relations, to obtain $\alpha_p \not\perp \alpha_r$, $\alpha_p \not\perp \alpha_q$ and $\alpha_q \perp \alpha_r$. But then condition (ii) of Definition 4.2.1 fails for $\overline{a}$, a contradiction. □

Lemma 4.2.4. *Suppose that* $\overline{a} = (\alpha_1, \dots, \alpha_n, \gamma, \beta_n, \dots, \beta_1)$ *is a braid sequence for some* $w \in W$. *Then for each* $1 \le q \le n$, *there is a sequence of type* $\overline{1}$ *and* $\overline{2}$ *moves carrying* $(\alpha_q, \dots, \alpha_n, \gamma, \beta_n, \dots, \beta_q)$ *to a braid sequence in which the roots* $\alpha_q, \gamma, \beta_q$ *appear consecutively.*

Proof. We argue by induction on n. The statement holds for $n = 1$, and also whenever $1 < q \le n$ by the inductive hypothesis. Suppose that $1 = q < n$. For the moment, assume the existence of an index $1 \le p < n$ such that $\alpha_p \perp \alpha_{p+1}$. If $p > 1$, then condition (ii) of Definition 4.2.1 gives $\alpha_{p-1} \perp \alpha_p$ or $\alpha_{p-1} \perp \alpha_{p+1}$. Applying a type $\overline{1}$ move in the latter case in order to swap α_p and α_{p+1} (note that $\beta_p \perp \beta_{p+1}$, by condition (iv) of Definition 4.2.1) and doing nothing in the former, we obtain a sequence in which the $(p-1)$th and pth entries are orthogonal. This sequence is a braid sequence for w, by Lemma 4.2.3. Continuing, we eventually reach a braid sequence in which the first two entries are orthogonal and the first entry is α_1 (the case $p = 1$). Apply a type $\overline{1}$ move to the outermost roots, and then invoke the inductive hypothesis.

If no p as in the previous paragraph exists, then apply a type $\overline{2}$ move to $\overline{a}$. The result is again a braid sequence for w, according to Lemma 4.2.3. By condition (iv) of Definition 4.2.1, $\alpha_{n-1} \perp \beta_n$, and we can proceed as above. The induction step is complete. □

4.3. **Correspondence between braid clusters and braid sequences.** The following proposition, together with Proposition 4.3.4, will enable us to move back and forth between braid sequences and braid clusters.

Proposition 4.3.1. *Let $w \in W$, let $\mathbf{i}$ be a reduced expression for w, and let $\overline{r}$ denote the root sequence of $\mathbf{i}$. Suppose that $\overline{r} = (\ldots, \alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1, \ldots)$, where $(\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$ is a braid sequence for w. Parse $\mathbf{i}$ as follows: if α_1 is the pth entry of $\overline{r}$, then write $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2\mathbf{i}_3$, where $\mathbf{i}_2$ has length $2n+1$ and $\mathbf{i}_3$ has length $p-1$. Then $\mathbf{i}_2$ is a braid cluster.*

Proof. Write $\mathbf{i}_2 = i_1 i_2 \cdots i_{2n+1}$. Let q be an integer with $1 \leq q \leq n$. By Lemma 4.2.4, there is a sequence of type $\overline{1}$ and $\overline{2}$ moves carrying $(\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$ to a braid sequence in which the roots $\alpha_q, \gamma, \beta_q$ appear consecutively. Fix such a sequence of moves, of minimum possible length. By Section 2.2, there is a corresponding sequence of type 1 and 2 moves, carrying $\mathbf{i}_2$ to a word $\mathbf{i}_2'$. By length-minimality, the moves have the effect that the nth and $(n+2)$th factors of $\mathbf{i}_2'$ equal, respectively, the qth and $(2n-q+2)$th factors of $\mathbf{i}_2$. By Section 2.2 again, we have $i_q = i_{2n-q+2}$.

So $\mathbf{i}_2 = i_1 \cdots i_n i_{n+1} i_n \cdots i_1$. Since $\mathbf{i}_2$ is reduced, for each $1 \leq q < n+1$, there is an index $q < r \leq n+1$ such that $m_{i_q i_r} = 3$. To see that r is uniquely determined by q, assume for a contradiction that there is a triple (q, r, s) such that $1 \leq q < r < s \leq n+1$ and $m_{i_q i_r} = m_{i_q i_s} = 3$. Among all such triples, choose one with q as large as possible. This forces $i_r \neq i_s$ (since $\mathbf{i}_2$ is reduced). Consider now a sequence of type 1 and 2 moves as in the first paragraph, transforming $\mathbf{i}_2$ in such a way that the qth and $(2n-q+2)$th factors of $\mathbf{i}_2$ are moved to positions n and $n+2$, with neither factor being involved in a long braid move. That this can be done implies there is no more than one $j \in I$ occurring (possibly more than once) in $i_{q+1} \cdots i_n i_{n+1} i_n \cdots i_{q+1}$ and satisfying $m_{i_q j} = 3$. We have reached a contradiction, since the distinct letters i_r, i_s both occur in $i_{q+1} \cdots i_n i_{n+1} i_n \cdots i_{q+1}$.

Thus, r is uniquely determined by q. It follows from this and the fact that $\mathbf{i}_2$ is reduced that the letters $i_1, \ldots, i_{n+1}$ are distinct. □

Definition 4.3.2. Let $w \in W$ be maximally clustered. We call a reduced expression for w *contracted* if its root sequence is contracted in the sense of Definition 3.3.6.

We are ready to present an analogue of Corollary 3.3.7 for reduced expressions.

Corollary 4.3.3. *Let $w \in W$ be maximally clustered. The following statements hold:*

(i) *Every reduced expression for w is short-braid equivalent to a contracted reduced expression.*

(ii) *If* $\mathbf{i}$ *is a contracted reduced expression for w, then* $\mathbf{i} = \mathbf{i}_0\mathbf{c}_1\mathbf{i}_1\mathbf{c}_2\mathbf{i}_2 \cdots \mathbf{c}_{\widetilde{N}(w)}\mathbf{i}_{\widetilde{N}(w)}$*, where each* $\mathbf{c}_q$ *is a braid cluster, of length* $2n_q + 1$ *say, and* $N(w) = \sum_q n_q$.

Proof. Let $\mathbf{j}$ be a reduced expression for w, and let $\overline{r}$ denote its root sequence. By Corollary 3.3.7, there is a sequence of short braid moves carrying $\overline{r}$ to a contracted root sequence, $\overline{r}'$. Let $\mathbf{j}'$ denote the reduced expression with root sequence $\overline{r}'$. By Section 2.2, $\mathbf{j}$ is short-braid equivalent to $\mathbf{j}'$. Statement (i) is proved.

For (ii), let $\overline{r}''$ denote the root sequence of $\mathbf{i}$, and let $C_1, \ldots, C_{\widetilde{N}(w)}$ be a complete, non-repeating list of the maximal sets of triples for w. For each $1 \leq q \leq \widetilde{N}(w)$, the roots in $\bigcup_{T \in C_q} T$ form a braid sequence in $\overline{r}''$, by Lemma 3.3.2, Proposition 3.3.8 and the fact that $\overline{r}''$ is contracted for C_q. These $\widetilde{N}(w)$ braid sequences are non-overlapping (by the paragraph following Definition 3.3.4) and account for all of the contractible triples of w. We now apply Proposition 4.3.1, once for each braid sequence, to complete the proof. □

The following proposition is the companion to Proposition 4.3.1.

Proposition 4.3.4. *Let $w \in W$, let* $\mathbf{i}$ *be a reduced expression for w, and let $\overline{r}$ denote the root sequence of* $\mathbf{i}$*. Suppose that* $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2\mathbf{i}_3$*, where* $\mathbf{i}_2 = i_1 i_2 \cdots i_n i_{n+1} i_n \cdots i_2 i_1$ *is a braid cluster. Denote the length of* $\mathbf{i}_3$ *by p, and let $\overline{a}$ denote the subsequence of $\overline{r}$ starting at entry $p+1$ and having $2n+1$ entries. Then $\overline{a}$ is a braid sequence for w.*

Proof. Write $\overline{a} = (\alpha_1, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_1)$, and let q be an integer with $1 \leq q \leq n$. By Lemma 4.1.3(ii), there is a sequence of type 1 and 2 moves carrying the braid cluster $i_q i_{q+1} \cdots i_n i_{n+1} i_n \cdots i_{q+1} i_q$ to a cluster in which the middle three factors are $i_q i_r i_q$ for some r with $m_{i_q i_r} = 3$. Fix such a sequence of moves, of minimum possible length. By Section 2.2, there is a corresponding sequence of type $\overline{1}$ and $\overline{2}$ moves carrying $(\alpha_q, \ldots, \alpha_n, \gamma, \beta_n, \ldots, \beta_q)$ to a sequence in which the middle three entries are $\alpha_q, \gamma, \beta_q$. By Section 2.2 again, we have $\alpha_q + \beta_q = \gamma$. Thus, $\{\alpha_q, \beta_q, \gamma\}$ is an inversion triple of w, contractible by type $\overline{1}$ and $\overline{2}$ moves.

For any $1 \leq p \neq q \leq n$, each of α_p, β_p is orthogonal to at most one of α_q, β_q: otherwise, for some $1 \leq p \neq q \leq n$, at least one of α_p, β_p would be orthogonal to the sum $\alpha_q + \beta_q = \gamma = \alpha_p + \beta_p$, which is impossible by Remark 3.1.5. Assume for a contradiction that for some $1 \leq p \neq q \leq n$, at least one of α_p, β_p is orthogonal to neither of α_q, β_q. For concreteness, assume that $\alpha_p \not\perp \alpha_q$ and $\alpha_p \not\perp \beta_q$ (the other case is similar). Observe that no type $\overline{1}$ or $\overline{2}$ move can reverse the relative order of α_p and α_q, or of α_p and β_q. If $p < q$, then, since α_q and β_q cannot simultaneously be on the same side of γ (recall Remark 3.1.4), no sequence of type $\overline{1}$ and $\overline{2}$ moves applied to $\overline{a}$ can result in a sequence in which the middle three terms are $\alpha_p, \gamma, \beta_p$. This contradicts the previous paragraph. If $p > q$, then α_p lies between α_q and β_q after any sequence of type $\overline{1}$ and $\overline{2}$ moves, again contradicting the previous paragraph.

We have so far shown that conditions (i) and (iv) of Definition 4.2.1 hold for $\overline{a}$. Turning to condition (ii), assume for a contradiction that there exist indices $1 \leq p < q < r \leq n$ such that $\alpha_q \perp \alpha_r$, $\alpha_p \not\perp \alpha_q$ and $\alpha_p \not\perp \alpha_r$. Then by condition (iv) we also have $\alpha_q \not\perp \beta_r$. After applying any sequence of type $\overline{1}$ and $\overline{2}$ moves to $\overline{a}$, we have a sequence in which α_p is to the left of α_q, which in turn is to the left of β_r. Also, α_p is to the left of α_r. Since α_r, β_r cannot simultaneously be on the same side of γ, the middle three terms cannot be $\alpha_p, \gamma, \beta_p$. This contradicts the first paragraph.

Finally, to see that condition (iii) of Definition 4.2.1 holds for $\overline{a}$, suppose that there exist indices $1 \leq p < q < r \leq n$ such that $\alpha_p \perp \alpha_r$, $\alpha_p \not\perp \alpha_q$ and $\alpha_q \not\perp \alpha_r$. Then we also have $\alpha_p \not\perp \beta_r$. As above, no sequence of type $\overline{1}$ and $\overline{2}$ moves can transform $\overline{a}$ into a sequence in which α_p is to the right of either α_r or β_r, and so $\{\alpha_p, \gamma, \beta_p\}$ is not contractible by type $\overline{1}$ and $\overline{2}$ moves. This last contradiction to the first paragraph completes the proof. □

We have the following converse to Corollary 4.3.3(ii).

Corollary 4.3.5. *Let $w \in W$. Suppose that w has a reduced expression* $\mathbf{i}$ *of the form* $\mathbf{i} = \mathbf{i}_0\mathbf{c}_1\mathbf{i}_1\mathbf{c}_2\mathbf{i}_2 \cdots \mathbf{c}_M\mathbf{i}_M$*, where each* $\mathbf{c}_q$ *is a braid cluster, of length* $2n_q + 1$ *say, and* $N(w) = \sum_q n_q$*. Then w is maximally clustered and* $\mathbf{i}$ *is contracted.*

Proof. Let $\overline{r}$ denote the root sequence of $\mathbf{i}$. By Proposition 4.3.4, we have braid sequences $\overline{a}_1, \ldots, \overline{a}_M$ in $\overline{r}$ corresponding to $\mathbf{c}_1, \ldots, \mathbf{c}_M$, respectively ($\overline{a}_M$ occurs to the left of $\overline{a}_{M-1}$ in $\overline{r}$, and so on). For each $1 \leq q \leq M$, the sequence $\overline{a}_q$ has $2n_q + 1$ entries, accounting for n_q contractible triples of w, and for any $1 \leq p \neq q \leq M$, no entry of $\overline{a}_p$ is an entry of $\overline{a}_q$. These facts, together with the equality $N(w) = \sum_q n_q$, ensure that the conclusion holds. □

5. A DELETION RESULT AND AN APPLICATION

Define a partial order $\leq$ on Φ as follows: given $\alpha, \beta \in \Phi$, we write $\alpha \leq \beta$ if the coefficients used to express $\beta - \alpha$ as a linear combination of simple roots are all nonnegative.

5.1. **Highest roots of inversion triples.**

The following technical result will be used repeatedly in the proof of Proposition 5.1.2, which in turn will be needed to prove the deletion theorem in Section 5.2. We continue to assume that W is of arbitrary simply laced type.

Proposition 5.1.1. [8, Lem. 2.1] *Let $w \in W$ and let $\overline{r}$ be a root sequence for w. Suppose that $\overline{r} = (\ldots, \alpha, \beta, \ldots)$, where $\alpha \not\perp \beta$ and $\alpha \not\leq \beta$. Then $\{\alpha - \beta, \alpha, \beta\}$ is a contractible triple of w.* □

The proposition below extends [8, Prop. 2.2] to the context of maximally clustered elements. The reader will recall from Remark 3.1.2 that there exist non-contractible inversion triples.

Proposition 5.1.2. *Let $w \in W$ be maximally clustered. Suppose that α is the highest root of an inversion triple of w. Then α is the highest root of a contractible inversion triple of w.*

Proof. Let $\overline{r}$ be a root sequence for w of the form $(\ldots, \alpha - \beta, \ldots, \alpha, \ldots, \beta, \ldots)$, where $\{\alpha - \beta, \alpha, \beta\}$ is an inversion triple of w. We argue by induction on the number of entries between α and β that α is the highest root of some contractible triple of w. If there are no entries between α and β, then the conclusion holds by Proposition 5.1.1 (we have $\alpha \not\perp \beta$ by Remark 3.1.5). Assume that $\overline{r} = (\ldots, \alpha, \delta_1, \ldots, \delta_n, \beta, \ldots)$, where $n \geq 1$. Either $\delta_n \perp \beta$, in which case a short braid move will place β closer to α, enabling us to apply the inductive hypothesis, or $\delta_n \not\perp \beta$. Assume the latter.

Suppose that $\{q : \delta_q \leq \beta\} = \emptyset$. Then $\{\delta_n - \beta, \delta_n, \beta\}$ is a contractible triple of w, by Proposition 5.1.1. For the moment, assume that $\delta_n - \beta \neq \alpha$. By Lemmas 3.3.1 and 3.3.3, a sequence of short braid moves, each exchanging $\delta_n - \beta$ with the entry directly to its right, carries $\overline{r}$ to a sequence of the form $(\ldots, \alpha, \ldots, \delta_n - \beta, \delta_n, \beta, \ldots)$, where every root between α and $\delta_n - \beta$ belongs to $\{\delta_1, \ldots, \delta_{n-1}\}$. A long braid move now places β closer to α than it was in $\overline{r}$. Observe that indeed we cannot have $\delta_n - \beta = \alpha$, since otherwise braid moves such as those just described would place α to the right of both $\alpha - \beta$ and β, contradicting Remark 3.1.4 (we also obtain contradictions to Remark 3.1.5 if $\delta_n - \beta = \alpha$).

Suppose now that $\{q : \delta_q \leq \beta\} \neq \emptyset$, and let p be the smallest element in this set. Move δ_p to the left in $\overline{r}$ as far as possible, using only short braid moves that involve δ_p. There are three possibilities to consider.

If δ_p can be moved past α, then in the resulting sequence α is closer to β than it was in $\overline{r}$, and we can apply the inductive hypothesis.

If δ_p can be moved past $\delta_1, \ldots, \delta_{p-1}$ but not α, then by Proposition 5.1.1, $\{\alpha - \delta_p, \alpha, \delta_p\}$ is a contractible triple of w (note that $\alpha \not\leq \delta_p$, since $\delta_p \leq \beta$ and α is the highest root of an inversion triple containing β).

Finally, suppose that there exists $1 \leq l < p$ such that $\delta_l \not\perp \delta_p$. Reindexing if necessary, we may assume that $l = p - 1$. By the definition of p, we have $\delta_{p-1} \not\leq \delta_p$. Hence, $\{\delta_{p-1} - \delta_p, \delta_{p-1}, \delta_p\}$ is a contractible triple of w, and there is a sequence of short braid moves, each exchanging $\delta_{p-1} - \delta_p$ with the root directly to its right, carrying $\overline{r}$ to a sequence $\overline{r}'$ in which the roots of this triple appear consecutively. Apply a long braid move to this triple in $\overline{r}'$, and denote the resulting sequence by $\overline{r}''$.

Note that we cannot have $\delta_{p-1} - \delta_p = \alpha$. Otherwise, on the one hand, α would be one of the two shorter roots of a contractible triple of w (namely, the triple in the previous paragraph). On the other hand, since α would be closer to β in $\overline{r}''$ than in $\overline{r}$, the inductive hypothesis would apply, so that α would be the highest root of some contractible triple of w. This would contradict the fact that w is maximally clustered.

Thus, we have $\overline{r}'' = (\ldots, \alpha, \ldots, \delta_p, \delta_{p-1}, \delta_{p-1} - \delta_p, \delta_{p+1}, \ldots, \delta_n, \beta, \ldots)$, where every entry between α and δ_p lies in $\{\delta_1, \ldots, \delta_{p-2}\}$. There are two cases to consider: (1)

$\delta_{p-1}-\delta_p$ occurs to the right of α in $\overline{r}$, in which case the number of roots between α and β is the same in $\overline{r}''$ as it is in $\overline{r}$, and (2) $\delta_{p-1}-\delta_p$ occurs to the left of α in $\overline{r}$, in which case the number of roots between α and β in $\overline{r}''$ is one greater than in $\overline{r}$.

Before examining these two cases, we first claim that (regardless of which of the cases holds) δ_p is orthogonal to every entry between α and δ_p in $\overline{r}''$, as well as to α. To see why, assume the contrary. Suppose that δ_p is orthogonal to every entry between α and δ_p, but not to α. Apply a sequence of short braid moves to $\overline{r}''$, each move exchanging δ_p with the entry directly to its left, to obtain the sequence $(\ldots,\alpha,\delta_p,\ldots,\delta_{p-1},\delta_{p-1}-\delta_p,\delta_{p+1},\ldots,\delta_n,\beta,\ldots)$. Recall that $\alpha\not\lneq\delta_p$ (the reason was given four paragraphs above). We are assuming here that $\alpha\not\perp\delta_p$, so Proposition 5.1.1 gives us that $\{\alpha-\delta_p,\alpha,\delta_p\}$ is a contractible triple of w. But there is another contractible triple of w having δ_p as one of its shorter roots, namely $\{\delta_{p-1}-\delta_p,\delta_{p-1},\delta_p\}$. The existence of two such triples contradicts the fact that w is maximally clustered. Suppose now that at least one of the entries between α and δ_p in $\overline{r}''$ is not orthogonal to δ_p, and let δ_k be the rightmost such entry (note that $1\le k\le p-2$). Apply a sequence of short braid moves to $\overline{r}''$, each exchanging δ_p with the entry directly to its left, to obtain $(\ldots,\alpha,\ldots,\delta_k,\delta_p,\ldots,\delta_{p-1},\delta_{p-1}-\delta_p,\delta_{p+1},\ldots,\delta_n,\beta,\ldots)$. We have $\delta_k\not\lneq\delta_p$, by the definition of p. We are assuming here that $\delta_k\not\perp\delta_p$, so Proposition 5.1.1 implies that $\{\delta_k-\delta_p,\delta_k,\delta_p\}$ is a contractible triple of w. Once again, δ_p is a shorter root in two different contractible triples of w (the other being $\{\delta_{p-1}-\delta_p,\delta_{p-1},\delta_p\}$), an impossibility. The claim is established.

Assume now that case (1) from two paragraphs above holds. By the above claim, δ_p can be commuted to the left in $\overline{r}''$ past α, resulting in a sequence in which α is closer to β than it was in $\overline{r}$. The inductive hypothesis can then be applied.

Assume at last that case (2) holds. Then $\delta_{p-1}-\delta_p$ is orthogonal to every entry between α and δ_p in $\overline{r}''$, as well as to α (indeed, $\delta_{p-1}-\delta_p$ was commuted past all these roots when we defined $\overline{r}'$). Hence, using the above claim again, $\delta_{p-1}=(\delta_{p-1}-\delta_p)+\delta_p$ is also orthogonal to all these entries. We can now move the three roots $\delta_p,\delta_{p-1},\delta_{p-1}-\delta_p$ to the left in $\overline{r}''$ past α, so that α is closer to β than it was in $\overline{r}$. ☐

Remark 5.1.3. Let $w\in W$ be maximally clustered. By the above proposition, $\widetilde{N}(w)$ equals the number of roots α such that α is the sum of two positive roots, each sent negative by w.

5.2. **Deletion of letters.** In [8, Prop. 2.4], Green and the author described the effect of deleting a letter from a contracted reduced expression for a freely braided element, assuming W is finite as well as simply laced. The following theorem generalizes this result. The proof is an adaptation of an argument of Fan [5, Thm. 1].

Theorem 5.2.1. *Assume that W is finite and simply laced. Let $w\in W$ be maximally clustered, and let $\mathbf{i}=i_1\cdots i_n$ be a contracted reduced expression for w. Then, deleting a single letter i_q from $\mathbf{i}$ results in a non-reduced word if and only if i_q is a middle letter of one of the $\widetilde{N}(w)$ braid clusters in $\mathbf{i}$.*

Proof. Write $\mathbf{i} = \mathbf{i}_0\mathbf{c}_1\mathbf{i}_1\mathbf{c}_2\mathbf{i}_2 \cdots \mathbf{c}_{\widetilde{N}(w)}\mathbf{i}_{\widetilde{N}(w)}$, where the word on the right is as in Corollary 4.3.3(ii). Suppose that the deletion of a letter i_q from $\mathbf{i}$ results in a non-reduced word. Then there exists $1 \leq p < q$ such that $i_{p+1} \cdots i_{q-1}i_{q+1} \cdots i_n$ is reduced and $i_p \cdots i_{q-1}i_{q+1} \cdots i_n$ is not. Letting $x = \phi(i_{p+1} \cdots i_{q-1})$ and $y = \phi(i_{q+1} \cdots i_n)$, we have $y^{-1}x^{-1}(\gamma_{i_p}) \in \Phi^-$ [9, Thm. 5.4]. Note that $x^{-1}(\gamma_{i_p}) \in \Phi^+$, since $i_p \cdots i_{q-1}$ is reduced.

Let $\alpha_1 = y^{-1}(x^{-1}(\gamma_{i_p}) + \gamma_{i_q})$, $\alpha_2 = -y^{-1}x^{-1}(\gamma_{i_p})$ and $\alpha_3 = y^{-1}(\gamma_{i_q})$. Clearly, $\alpha_1 + \alpha_2 = \alpha_3$. Let $\overline{r}$ denote the root sequence of $\mathbf{i}$. We shall show that α_1, α_2 and α_3 are entries of $\overline{r}$, so that $\{\alpha_1, \alpha_2, \alpha_3\}$ is an inversion triple of w.

It is clear that α_3 is the $(n - q + 1)$th entry of $\overline{r}$. Concerning α_2, from the first paragraph, we have $-y^{-1}x^{-1}(\gamma_{i_p}) \in \Phi^+$ and $y(-y^{-1}x^{-1}(\gamma_{i_p})) = -x^{-1}(\gamma_{i_p}) \in \Phi^-$. Thus, $\alpha_2 \in \Phi(y)$, so that α_2 occurs in the root sequence of $i_{q+1} \cdots i_n$, which is an initial segment of $\overline{r}$. It remains to show that α_1 is an entry of $\overline{r}$. Define $c = B(x^{-1}(\gamma_{i_p}), \gamma_{i_q})$, and suppose for now that $c = -1/2$. Then $\alpha_1 = y^{-1}(x^{-1}(\gamma_{i_p}) + \gamma_{i_q}) = y^{-1}(x^{-1}(\gamma_{i_p}) - 2c\gamma_{i_q}) = y^{-1}(s_{i_q}(x^{-1}(\gamma_{i_p}))) = y^{-1}s_{i_q}x^{-1}(\gamma_{i_p})$, which is the $(n - p + 1)$th entry of $\overline{r}$.

To see that c does in fact equal $-1/2$, note first that $x^{-1}(\gamma_{i_p}) \neq \gamma_{i_q}$ (the image of $x^{-1}(\gamma_{i_p})$ under y^{-1} is negative while the image of γ_{i_q} is positive). Since W is finite and since both of the roots $x^{-1}(\gamma_{i_p})$ and γ_{i_q} are positive, this implies $c \in \{-1/2, 0, 1/2\}$. If c were to equal 0, then $y^{-1}s_{i_q}x^{-1}(\gamma_{i_p})$, which is just the positive root α_1, would equal $y^{-1}x^{-1}(\gamma_{i_p})$, a negative root. If c were to equal $1/2$, then $y^{-1}s_{i_q}x^{-1}(\gamma_{i_p})$ would equal $y^{-1}(x^{-1}(\gamma_{i_p}) - \gamma_{i_q}) = y^{-1}x^{-1}(\gamma_{i_p}) - y^{-1}(\gamma_{i_q})$, which is again negative. Thus, $c = -1/2$, as desired.

By Proposition 5.1.2, α_3 is the highest root of a contractible triple of w. Now, the sequence $\overline{r}$ is contracted, and α_3 is its $(n-q+1)$th entry. Hence, by Proposition 4.3.1, i_q is the middle letter of one of the braid clusters $\mathbf{c}_1, \ldots, \mathbf{c}_{\widetilde{N}(w)}$.

On the other hand, it is clear that deleting a middle letter from any one of the above clusters results in a non-reduced word. □

Recall from Corollary 4.3.3(i) that every reduced expression for a maximally clustered element $w \in W$ is short-braid equivalent to a contracted expression. In view of this, Theorem 5.2.1 can be used to determine whether the deletion of a letter from an arbitrary reduced expression for w results in a reduced expression.

5.3. **Application to Schubert varieties.** The background material on Schubert varieties relevant to this section can be found in [3].

Assume now that our simply laced group W is the Weyl group of a semisimple, simply connected complex algebraic group $\mathcal{G}$ with fixed maximal torus $\mathcal{T}$ and Borel subgroup $\mathcal{B} \supset \mathcal{T}$. For each $w \in W$, let X_w denote the Schubert variety $\overline{\mathcal{B}w\mathcal{B}/\mathcal{B}}$ in the generalized flag variety $\mathcal{G}/\mathcal{B}$. Define, for each $w \in W$, a polynomial $P_w(t) = \sum_{v \preceq w} t^{\ell(v)}$, where $\preceq$ denotes the Bruhat–Chevalley partial order on W. The Poincaré polynomial for the cohomology ring of X_w is then given by $P_w(t^2)$.

Let $w \in W$. We define the *support* of w to be the set of all letters in I appearing in some (any) reduced expression for w. We denote this set by $\mathrm{supp}(w)$. If w is maximally clustered, then it has a contracted reduced expression as described in Corollary 4.3.3(ii). Consideration of such an expression reveals that $\#\text{supp}(w) \leq \ell(w) - N(w) \leq \ell(w) - \widetilde{N}(w)$.

We need the following basic fact about reduced expressions.

Proposition 5.3.1. [5, Lem. 2] *Let $w \in W$ and let $i_1 \cdots i_n$ be a reduced expression for w. For each $1 \leq q \leq n$, let $w_q = \phi(i_1 \cdots \widehat{i_q} \cdots i_n)$, where the hat indicates omission. Then the w_q, $1 \leq q \leq n$, are distinct. Furthermore, the set $\{w_q : 1 \leq q \leq \ell(w)\}$ is independent of the choice of reduced expression for w.* □

Let $v, w \in W$ and let $i_1 \cdots i_n$ be a reduced expression for w. Then $v \preceq w$ if and only if v is represented by a (possibly empty) product of the form $i_{q_1} \cdots i_{q_k}$ where $1 \leq q_1 < \cdots < q_k \leq n$ (see, e.g., [9, Thm. 5.10]).

Theorem 5.3.2. *Let $w \in W$ be maximally clustered. Then the Schubert variety X_w is smooth if and only if $\#\text{supp}(w) = \#\{\gamma \in \Phi(w) : \gamma \neq \alpha + \beta \text{ for any } \alpha, \beta \in \Phi(w)\}$.*

Proof. First note that $\#\{\gamma \in \Phi(w) : \gamma \neq \alpha + \beta \text{ for any } \alpha, \beta \in \Phi(w)\} = \ell(w) - \widetilde{N}(w)$, by Remark 5.1.3 and the equality $\#\Phi(w) = \ell(w)$.

By Theorem 5.2.1, Proposition 5.3.1 and the above characterization of $\preceq$, there are exactly $\ell(w) - \widetilde{N}(w)$ elements $v \in W$ with $\ell(v) = \ell(w) - 1$ and $v \preceq w$. If X_w is smooth, then the coefficients of $P_w(t)$ form a palindromic sequence, by Poincaré duality, and it follows that there are exactly $\ell(w) - \widetilde{N}(w)$ elements $v \in W$ of length 1 with $v \preceq w$, i.e., $\#\text{supp}(w) = \ell(w) - \widetilde{N}(w)$.

Conversely, if $\#\text{supp}(w) = \ell(w) - \widetilde{N}(w)$, then the inequalities preceding Proposition 5.3.1 give $\widetilde{N}(w) = N(w)$. Thus, w is freely braided, and we can now invoke the "if" part of [8, Thm. 3.2] to complete the proof. □

The above argument shows that all maximally clustered, non-freely braided elements of W correspond to singular Schubert varieties. This is consistent with the discussion at the end of Section 3.2, where it was observed that every maximally clustered, non-freely braided permutation contains the pattern 4231. It is known [11] that in type A, a Schubert variety X_w cannot be smooth if w contains this pattern or 3412.

We remark that another possible approach to proving the above theorem would be to use the results in the recent paper [2].

Generating functions for freely braided $w \in W$ such that X_w is smooth are given in [8, Thm. 4.2]; such w are said to be "content maximal" there.

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