THE PARABOLIC MAP

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Abstract. In this paper we study certain aspects of Bruhat order on Coxeter groups. Let W be a Coxeter group, and let $\mathcal{P}$ denote the set of parabolic subgroups of W. We construct a map $m : W \times \mathcal{P} \to W$ and examine its first properties. We prove the following characterization: w is the longest element of a parabolic subgroup of W if and only if for any $v \in W$, there exists a unique maximal element (with respect to Bruhat order) less than or equal to both v and w.

1. Introduction

Let W be a Coxeter group with Coxeter graph Γ. A basic reference for Coxeter groups is [2]. Denote the set of Coxeter generators by S; these are in one-to-one correspondence with the nodes of Γ. The group W is generated by elements $s \in S$ with the relations $s^2 = 1$ for all $s \in S$ and $(st)^{m_{st}} = 1$ for $s, t \in S$, where $m_{st} = m_{ts}$ is an integer which depends on the weight of the edge connecting the nodes corresponding to s and t. Every element w can be written as a product of generators $s_1 \cdots s_n$, $s_k \in S$. When n is minimal, we call this product *reduced* and define the length of w, denoted $l(w)$, to be n.

The Bruhat order is a natural partial order on W which arose from geometric properties of the decomposition of the flag manifold into affine cells (known as the Bruhat decomposition) (see Section 5). Suppose $s_1 \cdots s_n$ is a reduced expression for w. Then $w' \leq w$ in the Bruhat order if and only if $w' = s_{i_1} \cdots s_{i_k}$ for some $1 \leq i_1 < \cdots < i_k \leq n$. In fact, by [6, Theorems 1.7, 5.10], we know that if $w_1 < w_2$, then for any reduced expression for w_2, there exists a substring of that reduced expression which is reduced and whose product is w_1.

The main question we address in this paper is: Given $v, w \in W$, is there a unique maximal element u less than or equal to both v and w? Our main result is Theorem 4.1 which characterizes the longest elements of parabolics (when they exist) in terms of this question. We also investigate the map $m : W \times \mathcal{P} \to W$ which assigns to each $w \in W$ and parabolic subgroup a unique maximal element as described above.

Some motivation for this work comes from the study of Bruhat intervals of the form $[\hat{0}, w] \equiv \{v \in W \mid v \leq w\}$. It is known [5, 1] that certain such intervals contain a subposet on all vertices that is the product of a chain with a smaller interval. The

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map m plays an implicit role in this subposet factorization. Moreover, if w indexes a rationally smooth Schubert variety of type A, B, or C, then this factorization procedure can be applied recursively, resulting in a subposet of $[\hat{0}, w]$ that is a product of chains. This implies that the rank generating function $p_w(t) = \sum_{v \leq w} t^{l(v)}$ factors into a product of polynomials of the form $\frac{t^k - 1}{t-1}$.

2. The map m

Let $\mathcal{P}$ denote the set of parabolic subgroups of W, that is, subgroups generated by subsets of S.

Lemma 2.1. *Let $s_1, \ldots, s_N$ be any sequence of generators from S (repeated elements allowed) and fix a parabolic subgroup P. Let P' denote the subset of elements in P which can be written as a product $s_{n_1} \cdots s_{n_l}$ with $1 \leq n_1 < \cdots < n_l \leq N$. Then P' has a unique maximal element with respect to Bruhat order.*

Proof. We prove the lemma by induction on N. Suppose $N = 1$. Then $P' = \{1\}$ if $s_1 \notin P$ and $P' = \{1, s_1\}$ if $s_1 \in P$. By inspection, the conclusion of the lemma is true.

Now suppose the lemma is true for all sequences of length less than N. Let $s_1, \ldots, s_N$ be a sequence of generators from S, and define P' with respect to this sequence as in the statement of the lemma. (Note that P' is necessarily a finite set.) Similarly, define Q' with respect to the sequence $s_1, \ldots, s_{N-1}$. Note that $Q' \subset P'$. By induction, there is a unique maximal element $q' \in Q'$.

We claim that either q' or $q's_N$ is the unique maximal element of P' depending on whether $q's_N < q'$ or $q' < q's_N$, respectively. (Note that q' and $q's_N$ are comparable.) To see this, let v be any element in P'. Say $v = s_{n_1} \cdots s_{n_l}$ with $1 \leq n_1 < \cdots < n_l \leq N$.

We consider two cases: either $n_l < N$ or $n_l = N$.

If $n_l < N$, then $v \in Q'$ so that $v \leq q'$.

If $n_l = N$, then $vs_N \in Q'$, so that $vs_N \leq q'$. It is a known property of Bruhat order [6, Proposition 5.9] that, given $s \in S$ and $x, y \in W$ such that $x \leq y$, either $xs \leq y$ or $xs \leq ys$ (or both). Hence, $v = (vs_N)s_N \leq q'$ or $v = (vs_N)s_N \leq q's_N$.

Since q' and $q's_N$ are comparable, the lemma follows. □

Theorem 2.2. *For any parabolic subgroup P and any $w \in W$, there exists a unique maximal element less than or equal to w in P (with respect to Bruhat order).*

Proof. Pick a reduced expression $r_1 \cdots r_{l(w)}$ for w. Let r_{n_k}, $k = 1, \ldots, l$, $n_1 < \cdots < n_l$, be those generators in the reduced expression such that $r_{n_k} \in P$ and then associate to the sequence $s_k = r_{n_k}$ the set P' as in Lemma 2.1.

We claim that P' comprises the set of elements $v \in P$ such that $v \leq w$ (regardless of the choice of reduced expression for w). To see this, first note that since P is generated by a subset of S, any reduced expression for an element $v \in P$ involves only generators in P. Recall that if $w_1 < w_2$, then for any reduced expression for

w_2, there exists a substring of that reduced expression which is reduced and whose product is w_1. Therefore, if $v \leq w$ and $v \in P$, then for any reduced expression for w (in particular, for the reduced expression $r_1 \cdots r_{l(w)}$), there exists a reduced subexpression equal to v, and this reduced subexpression involves only generators belonging to P. Conversely, any $v \in P'$ satisfies $v \leq w$ and $v \in P$ by construction.

By Lemma 2.1, there is a unique maximal element v' in P', as desired. □

Theorem 2.2 enables us to define a map $m : W \times \mathcal{P} \to W$ as follows. Given $w \in W$ and $P \in \mathcal{P}$, let $m(w, P)$ denote the unique maximal element in P less than or equal to w.

We conclude this section by giving a method to compute $m(w, P)$.

Let $\mathcal{A}$ be the algebra (with unit) over $\mathbb{Z}$ with generators X_s, $s \in S$, which satisfy the braid relations $X_s X_t X_s \cdots = X_t X_s X_t \cdots$ (each side has m_{st} terms) and the relations $X_s^2 = X_s$, $s \in S$. (This is isomorphic to the specialization at zero of the Iwahori-Hecke algebra associated to W.)

The algebra $\mathcal{A}$ has a basis X_w parameterized by $w \in W$, where we can take $X_w = X_{s_1} \cdots X_{s_n}$ for any reduced expression $s_1 \cdots s_n$ for w.

Let $w = s_1 \cdots s_n$ be reduced, and suppose that $s_{i_1}, \ldots, s_{i_l}$ is the sequence of all the generators in the reduced expression which are in P, taken in order. Then $X_{m(w,P)} = X_{s_{i_1}} \cdots X_{s_{i_l}}$. We omit details of the proof which proceeds by induction on $l(w)$ following lines close to that of the proof of Lemma 2.1.

3. Properties of the map m

Order the elements of $\mathcal{P}$ by inclusion. For $w \in W$, denote by $c(w)$ the set of generators in S which appear in some (or any) reduced expression for w. Also, for any $P \in \mathcal{P}$, define $P^{op} \in \mathcal{P}$ to be the parabolic subgroup generated by $S \cap \bar{P}$ (where $\bar{P}$ is set-theoretic complement of P in W).

Theorem 3.1. *The map $m : W \times \mathcal{P} \to W$ enjoys the following properties.*

1. *We have $m(w, P) \in P$. If $w \in P$, then $m(w, P) = w$.*
2. *If we fix $w \in W$, the map $m(w, \cdot) : \mathcal{P} \to W$ is order preserving. Furthermore, the fiber of $m(w, \cdot)$ over $m(w, P)$ consists of those $P' \in \mathcal{P}$ such that $P' \cap c(w) = P \cap c(w)$. In particular, the fiber is naturally isomorphic to the lattice of subsets of the set $S \setminus c(w)$.*
3. *If we fix $P \in \mathcal{P}$, the map $m(\cdot, P) : W \to P$ is order preserving. Furthermore, the map $m(\cdot, P) : W \to P$ factors through $P^{op} \backslash W / P^{op}$.*
4. *If $P_2 \subset P_1$, where P_1, $P_2 \in \mathcal{P}$, then $m(m(w, P_1), P_2) = m(w, P_2)$.*

Proof. Property 1 follows from the definition.

For Property 2, fix $w \in W$. Let P_1, $P_2 \in \mathcal{P}$. By Lemma 2.1, if $P_1 \cap c(w) = P_2 \cap c(w)$, then $m(w, P_1) = m(w, P_2)$. On the other hand, if there exists $s \in P_1 \cap c(w)$ such that $s \notin P_2 \cap c(w)$, then by construction we see that $s \leq m(w, P_1)$, whereas $s \not\leq m(w, P_2)$.

Hence $m(w, P_1) \neq m(w, P_2)$, which proves the fiber over $m(w, P)$ consists of all $P' \in \mathcal{P}$ such that $P' \cap c(w) = P \cap c(w)$.

Now, if $P_1 \subset P_2$, then every element less than or equal to w in P_1 is less than or equal to $m(w, P_2)$ by definition. In particular, $m(w, P_1) \leq m(w, P_2)$. Property 2 follows.

For Property 3, let x denote the minimal length coset representative of some left coset of P^{op} in W. Any element of xP^{op} can be written uniquely as xq where $q \in P^{op}$ and $l(xq) = l(x) + l(q)$ [6, Section 5.12]. By the construction in Theorem 2.2, $m(xq, P) = m(x, P)$. This implies that $m(\cdot, P)$ factors through W/P^{op}. A symmetric argument implies that $m(\cdot, P)$ is also left P^{op} invariant. We conclude that $m(\cdot, P)$ is P^{op} bi-invariant.

Now suppose $w_1 \leq w_2$. Then $m(w_1, P) \leq w_1$ so that $m(w_1, P) \leq w_2$ by transitivity. By construction, $m(w_2, P)$ is the unique maximal element in P less than or equal to w_2. It follows that $m(w_1, P) \leq m(w_2, P)$. Property 3 follows.

For Property 4, we note that $m(w, P_2) \in P_2 \subset P_1$, so that by definition, $m(w, P_2) \leq m(w, P_1)$. Hence $m(w, P_2) \leq m(m(w, P_1), P_2)$. On the other hand, every $v \in P_2$ such that $v \leq w$ satisfies $v \leq m(w, P_2)$. Since $m(m(w, P_1), P_2) \leq m(w, P_1) \leq w$ and $m(m(w, P_1), P_2) \in P_2$, we see that $m(m(w, P_1), P_2) \leq m(w, P_2)$. Property 4 follows. □

We remark that the induced map $m(\cdot, P) : P^{op}\backslash W/P^{op} \to P$, while always surjective, is not generally injective. For instance, let W be of type A_3 and label the (Coxeter) generators S by x, y, and z so that $xz = zx$. Let $P = \langle x, z\rangle$. Then $m(xyxzyx, P) = m(xz, P) = xz$, but xz and $xyxzyx$ are not in the same P^{op} double coset.

We also note that the map $m(\cdot, P) : W \to P$ is a group homomorphism if and only if the subgraph of Γ corresponding to $P \cap S$ constitutes a union of some connected components of Γ. (This is equivalent to P being a normal subgroup of W.)

4. Characterization of longest elements of parabolics

In this section, we prove what is essentially a converse to Theorem 2.2.

Theorem 4.1. *Let $w \in W$. Then w has the property that for all $v \in W$, the set $\{u \in W \mid u \leq w \text{ and } u \leq v\}$ has a unique maximal element if and only if w is the longest element of some parabolic subgroup of W.*

Proof. Let $P \in \mathcal{P}$ and suppose P has a unique maximal length element w_P. In this case, every $p \in P$ satisfies $p \leq w_P$, and if $p \leq w_P$, then $p \in P$. For this reason, and because of Theorem 2.2, it suffices to show that if w has the property that for all $v \in W$, there exists a unique maximal element less than or equal to w and v, then w is the the longest element of a parabolic subgroup of W. So let w have this property.

Let $P \in \mathcal{P}$ be generated by $c(w)$. Note that $w \in P$. We show that every $p \in P$ satisfies $p \leq w$ by induction on $l(p)$.

If $l(p) = 0$, then $p = 1$ and $1 \leq w$. If $l(p) = 1$, then $p \leq w$ since $p \in c(w)$.

Now, suppose that $N > 1$, and $q \leq w$ for all $q \in P$ such that $l(q) < N$. Let $p \in P$ have length N. Let $p = s_1 \cdots s_N$ be a reduced expression for p. By induction, both $p_N = s_1 \cdots s_{N-1}$ and $p_1 = s_2 \cdots s_N$ are less than or equal to w. Note that $p_1 \neq p_N$ since $p_1 = p_N$ implies that $p = p_N s_N = p_1 s_N = s_2 \cdots s_{N-1}$, which is a contradiction since the length of $s_2 \cdots s_{N-1}$ is $l(p) - 2$. Furthermore, since the lengths of both p_1 and p_N are $N - 1$, p_1 and p_N must be incomparable with respect to Bruhat order. By assumption, there exists a unique maximal v less than or equal to p and w. This means that p_1 and p_N are both less than or equal to v, since they are both less than or equal to p and w. But the only element less than or equal to p and greater than or equal to both p_1 and p_N is p itself. Hence $v = p$, and the induction step is proven. On the other hand, we know that $w \in P$. This is possible only if P is finite and w is the unique longest element in P. □

5. A geometric interpretation

Let G be a complex, reductive Lie group. Fix a Borel subgroup B of G. Let W be the associated Weyl group. Then the orbits O_w of B acting on G/B by left translation are affine spaces parameterized by W. The closures of these orbits, denoted S_w, are known as Schubert varieties. Chevalley [4] showed that $S_w = \cup_{v \leq w} O_v$, where $\leq$ denotes the Bruhat order (or Bruhat–Chevalley order). (This can be taken to be the definition of Bruhat order.)

Our results in this special case can be restated as follows. Fix $w \in W$. Then w is the longest element of a parabolic if and only if for any $v \in W$, we have $S_w \cap S_v = S_{v'}$ for some $v' \in W$. We remark that this intersection does not correspond to homological intersection, which involves placing the varieties into general position with respect to each other.

Tom Braden has given a geometric proof of the fact that if w is the longest element of a parabolic then for any $v \in W$, there exists $v' \in W$ such that $S_w \cap S_v = S_{v'}$.

The Poincaré polynomial of the cohomology ring of a Schubert variety S_w is $p_w(t^2)$, where $p_w(t)$ denotes the rank generating function of the Bruhat interval $[\hat{0}, w]$. Let S_w be a rationally smooth Schubert variety of type A, B or C. The factorization of $p_w(t^2)$ mentioned in the Introduction suggests the possibility that S_w may be exhibited as a simple kind of fiber bundle involving projective spaces.

It follows from the work of Carrell and Peterson [3] that if the Poincaré polynomial of a Schubert variety S_w has a factorization as described above, then S_w is rationally smooth. It may be conjectured that for any rationally smooth Schubert variety, the corresponding Bruhat interval contains a product of chains as a subposet on all vertices.

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