

MATCHINGS AND CANONICAL FORMS FOR SYMMETRIC TENSORS

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ABSTRACT. Let V be a q -dimensional vector space. Fix a set B of $q(q-1)$ monomials in $S^p(V)$ of the form x^I where $i_k > 0$ for all k . The generic element of $S^p(V)$ is conjugate under a suitable linear transformation to an element with support off of B . We prove this by showing the existence of a perfect matching with a unique weight in a certain weighted bipartite graph. Such a perfect matching corresponds to the non-vanishing of an appropriate determinant.

1. INTRODUCTION

In 1916, E.K. Wakeford described a technique for the study of canonical forms for symmetric tensors. In particular, he considered the question of which sets of monomials are removable from a generic homogeneous polynomial through a linear change in its variables.

Let V be a q -dimensional vector space. In this paper, we show that if p is sufficiently large, virtually any set of $q(q-1)$ monomials are removable from a generic homogeneous polynomial of degree p under a linear change in its variables (see theorem 4.4.2 for a precise statement). Furthermore, at most $q(q-1)$ monomials are so removable.

To achieve this result, we needed some results on perfect matchings which we describe in section 2. In section 3, we describe *apolarity*, a notion dating back to Wakeford's time. Apolarity is a tool which helps to organize our proofs by allowing us to avoid manipulations with various Jacobian-type matrices. Finally, in section 4, we state and prove our main results on canonical forms.

We wish to thank Gian-Carlo Rota for introducing us to this problem.

2. MATCHING THEORY

In this section, D will denote a finite subset of \mathbb{Z}^q which does not contain the zero vector and B will denote a subset of \mathbb{Z}^q with $|B| = |D|$.

We denote by $(\cdot, \cdot) : \mathbb{Z}^q \times \mathbb{Z}^q \rightarrow \mathbb{Z}$ a positive definite, non-degenerate bilinear form. Any geometric notions used are with respect to this bilinear form.

We remark that all the results are valid if we replace \mathbb{Z} with \mathbb{Q} or \mathbb{R} .

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2.1. An application of Hall's marriage theorem.

Definition 2.1.1. A *perfect matching* is a bijection $f : B \rightarrow D$ such that $b + f(b) \notin B$ for all $b \in B$.

For any point $v \in \mathbb{Z}^q$ and any $w \in \mathbb{Z}^q$ with $w \neq 0$, define

$$H(v, w) = \{x \in \mathbb{Z}^q \mid (x - v, w) \leq 0\}.$$

Let $S \subseteq B$ and $d \in D$. We say that S *likes* d if there exists $b \in S$ such that $b + d \notin B$. We say that S *hyperplane-likes* d if there exists $b \in S$ such that $b + d \notin B$ and $S \subseteq H(b, d)$. If $S = \{b\}$, we shall also use the terminology b likes d , or b hyperplane-likes d . Finally, denote by $W(S, d) \subseteq B$ the set of $b \in S$ such that $S \subseteq H(b, d)$. Notice that by definition of $W(S, d)$, we can have $b \in W(S, d)$ such that $b + d \in B$. Also note that if $S \neq \emptyset$, then $W(S, d) \neq \emptyset$, because S is finite.

Proposition 2.1.2. *Given $S \subseteq B$, let $L \subseteq D$ denote those $d \in D$ which are hyperplane-liked by S . Then $|L| \geq |S|$.*

Proof. For each $d \in D \setminus L$, pick any element in $W(S, d)$ and denote it b_d . Observe that $b_d + d \in B \setminus S$. We claim that if $b_d + d = b_c + c$, then $d = c$. Let $p = b_d + d$. Observe that b_d is the unique closest point to p in S since $S \subseteq H(b_d, d)$. Similarly, b_c is the unique closest point to p in S . Therefore $b_d = b_c$ and so $d = c$.

From this we have that $|\{b_d + d\}_{d \in D \setminus L}| = |D \setminus L|$. Consequently, $|B| \geq |S| + |D| - |L|$ so that $|L| \geq |S|$, as desired.

From proposition 2.1.2, we have the following direct application of Hall's Marriage Theorem.

Corollary 2.1.3. *There exists a perfect matching.*

Remark 2.1.4. We wish to point out that Bjorn Poonen has found a different proof of corollary 2.1.3 which exploits the structure of \mathbb{Z}^q as an ordered group.

2.2. A matching algorithm. In this section, we shall describe a simple algorithm which produces a perfect matching whenever D has the property that if $v, cv \in D$ with $c > 0$ then $c = 1$. This algorithm will yield a matching in $O(|D|^2)$ time.

We find it convenient to refer to elements of B as “balls” and elements of D as “arrows”. Also, we shall speak of “assigning” arrows to balls.

Suppose we have assigned a number of arrows to various balls such that no ball is assigned more than one arrow and each ball which is assigned an arrow likes the arrow which is assigned to it.

Take an unassigned arrow $d \in D$. Pick any unassigned ball $b \in B$.

If b likes d , then assign d to b .

If b does not like d then let k be the smallest positive integer for which $b + kd \in B$ and $b + kd$ likes d . If $b + kd$ is unassigned, assign it d . Otherwise, take away the arrow d' which was assigned to it, and reassign $b + kd$ the arrow d .

If b likes d' , assign d' to b . Otherwise, repeat the procedure in the previous paragraph with d' in place of d . Notice that this procedure cannot be repeated indefinitely because the balls $b + kd$ will never be observed twice since we are assuming that if $d, cd \in D$ with $c > 0$, then $c = 1$.

In this way, we can add, one by one, all the arrows in D and produce a perfect matching.

2.3. Acyclic matchings.

Definition 2.3.1. For any perfect matching f , we define a map $m_f : \mathbb{Z}^q \rightarrow \mathbb{Z}$ by letting $m_f(v) = \#\{b \in B \mid b + f(b) = v\}$.

Definition 2.3.2. An *acyclic* matching is a perfect matching f with the property that for any perfect matching g with $m_f = m_g$, we have $f = g$.

Our goal is to show that acyclic matchings exist.

Definition 2.3.3. A *hyperplane chain* C of length l is a sequence of pairs $(b_i, d_i) \in B \times D$ for $i = 1, \dots, l$ such that the following four properties hold:

- (1) The b_i are all distinct;
- (2) The d_i are all distinct;
- (3) b_i likes d_i ;
- (4) $B \setminus \{b_1, \dots, b_{i-1}\} \subseteq H(b_i, d_i)$.

Given a hyperplane chain $C = \{(b_i, d_i)\}$ of length l , we define $B(C) = \{b_1, \dots, b_l\}$ and $D(C) = \{d_1, \dots, d_l\}$. Note that by definition, $|B(C)| = |D(C)| = l$. We also define $B^i(C) = b_i$ and $D^i(C) = d_i$.

Remark 2.3.4. Observe that if $B \setminus \{b_1, \dots, b_{i-1}\} \subseteq H(B^j(C), D^j(C))$ for some $i < j$, we can obtain a new hyperplane chain by rearranging the terms in the hyperplane chain according to the cyclic permutation $(i \ i+1 \ i+2 \ \dots \ j)$.

Proposition 2.3.5. *Let $T \subseteq D$. There exists a hyperplane chain C of length $|T|$ such that $T = D(C)$.*

Proof. Let P_l denote the statement: Given a hyperplane chain C of length l and any $d \in D \setminus D(C)$, there exists a hyperplane chain C' of length $l + 1$ such that $D(C') = D(C) \cup \{d\}$ and $B(C) \subseteq B(C')$.

We shall prove the statement P_l for $0 \leq l < |B|$ by induction on l .

First we show that P_0 is true. Pick any $b \in W(B, d)$. Then (b, d) is a hyperplane chain of length 1. This proves P_0 .

Now assume that P_s is true for all $s < l < |D|$. We show that P_l is true.

Let $U = B \setminus B(C)$. By proposition 2.1.2, U hyperplane-likes at least $|B| - l$ elements in D . Therefore, U must hyperplane-like at least one vector in $D(C) \cup \{d\}$.

Let $C_0 = C$ and let $d_0 = d$.

We construct, by induction on i , vectors d_i and hyperplane chains C_i of length l such that $B(C) = B(C_i)$ and, furthermore, U does not hyperplane-like any $D^j(C_i)$ for $1 \leq j \leq i$.

Assume C_i and d_i have been defined and $i < l$.

If U does not hyperplane-like d_i , we construct a hyperplane chain C_{i+1} of length l and d_{i+1} as follows. By induction hypothesis P_i , there exists a hyperplane chain C^* of length $i + 1$ which extends the hyperplane chain of length i consisting of the pairs $\{(B^j(C_i), D^j(C_i))\}_{j=1, \dots, i}$ by the element d_i . Since U does not hyperplane-like d_i nor any of $D^j(C_i)$ for $1 \leq j \leq i$, we must have $B(C^*) \subseteq B(C)$. Consequently, $B(C^*) = \{B^j(C_i)\}_{j=1, \dots, i} \cup \{B^m(C_i)\}$ for some $i < m \leq l$.

Define C_{i+1} as follows. Let $B^j(C_{i+1}) = B^j(C^*)$ and $D^j(C_{i+1}) = D^j(C^*)$ for $1 \leq j \leq i + 1$. Let the sequence $B^{i+2}(C_{i+1}), \dots, B^l(C_{i+1})$ equal $B^{i+1}(C_i), \dots, B^l(C_i)$ with the term $B^m(C_i)$ removed. Similarly, let the sequence $D^{i+2}(C_{i+1}), \dots, D^l(C_{i+1})$ equal the sequence $D^{i+1}(C_i), \dots, D^l(C_i)$ with the term $D^m(C_i)$ removed.

By construction, C_{i+1} is a hyperplane chain of length l . (Recall the remark 2.3.4.)

Finally, let $d_{i+1} = D^m(C_i)$.

Note that only finitely many C_i can be constructed because the d_i constitute a set of distinct elements from $D(C) \cup \{d\}$ and we have observed that U must hyperplane-like at least one element in $D(C) \cup \{d\}$. Let M denote the largest integer for which C_M is defined (we might have $M = 0$).

Then, d_M is hyperplane-liked by U (otherwise C_{M+1} would be defined). In other words, there exists a $b \in U$ which likes d_M and, furthermore, $U \subseteq H(b, d_M)$. We now define C' by letting $B^i(C') = B^i(C_M)$ and $D^i(C') = D^i(C_M)$ for $1 \leq i \leq l$ and then setting $B^{l+1}(C') = b$ and $D^{l+1}(C') = d_M$.

The construction of C' shows that the statement P_l is true.

By induction P_l is true for all $0 \leq l < |D|$.

As we noted in proving P_0 , every element of D belongs to some hyperplane chain of length 1. By successively applying P_l , the proposition follows.

We are now ready to prove the main result of section 1.

Theorem 2.3.6. *Assume that $(d, d) = (d', d')$ for all $d, d' \in D$. Then there exists an acyclic matching.*

Proof. By proposition 2.3.5, there exists a hyperplane chain C of length $|B|$. Define $f : B \rightarrow D$ by $f(b_i) = d_i$ for $1 \leq i \leq |B|$. We claim f is an acyclic matching.

Let g be any other perfect matching such that $m_f = m_g$.

We prove by induction on i that $f(b_i) = g(b_i)$.

When $i = 1$, we have that $B \subseteq H(b_1, d_1)$. Since b_1 is the unique closest point to $b_1 + d_1$ in $H(b_1, d_1)$, we have that $m_f(b_1) = 1$ and necessarily, $g(b_1) = d_1$.

Now assume that $f(b_i) = g(b_i)$ for all $i < k$. We show that $f(b_k) = g(b_k)$.

Since b_k is the unique closest element of $B \setminus \{b_1, \dots, b_{k-1}\} \subseteq H(b_k, d_k)$ to $b_k + d_k$, we have $m_f(b_k + d_k) = 1 + \#\{1 \leq j \leq k - 1 \mid b_j + d_j = b_k + d_k\} = m_g(b_k + d_k)$. We must therefore have $g(b_k) = d_k$.

Thus, $f = g$ and the theorem follows.

Remark 2.3.7. The proof of theorem 2.3.6 goes through under the weaker hypothesis that $(d' - d, d) \leq 0$ with equality if and only if $d = d'$.

Remark 2.3.8. Under the hypothesis of theorem 2.3.6, the resulting perfect matching f has the property that m_f takes values in $\{0, 1\}$.

Remark 2.3.9. It seems possible that theorem 2.3.6 is true without any hypothesis on D . The above method of proof cannot apply to the general case, however. For general D , it is possible to have a hyperplane chain which does not yield an acyclic matching. For instance, if our lattice is \mathbb{Z} and $D = \{1, 2, 3, 4\}$, every perfect matching comes from a hyperplane chain. Let $B = \{0, 1, 3, 5\}$. Define the following perfect matchings:

$$\begin{aligned} f : 0 &\mapsto 4, 1 \mapsto 1, 3 \mapsto 3, \text{ and } 5 \mapsto 2; \\ g : 0 &\mapsto 2, 1 \mapsto 3, 3 \mapsto 4, \text{ and } 5 \mapsto 1. \end{aligned}$$

Then, $m_f = m_g$.

In the sequel, we shall be primarily interested in the case where D is the set of roots of the root system of type A .

3. APOLARITY AND CANONICAL FORMS

Let V be a q -dimensional vector space over the complex field \mathbb{C} and consider the symmetric algebra over V :

$$S(V) = \bigoplus_{p \geq 0} S^p(V),$$

where $S^p(V)$ are the symmetric tensors of degree p . Since $S^p(V)$ is finite dimensional over \mathbb{C} , we may endow it with the Euclidean topology. We say that a property P holds *generically* in $S^p(V)$ if P holds for every element of some dense subset of $S^p(V)$.

We are concerned with finding canonical expressions for elements of $S^p(V)$. The canonical forms we seek will be generic in the sense that each represents some dense subset of $S^p(V)$.

3.1. The apolar bilinear form. Fix a basis x_1, \dots, x_q of V , and let u_1, \dots, u_q be the corresponding dual basis of V^* . Let $I = (i_1, \dots, i_q) \in \mathbb{Z}^q$. We define $x^I = x_1^{i_1} \cdots x_q^{i_q}$ and (provided $i_k \geq 0$) $I! = i_1! \cdots i_q!$. Also, define $e_r = (\delta_{1,r}, \dots, \delta_{q,r}) \in \mathbb{Z}^q$.

Following Ehrenborg and Rota [1], we define the *apolar bilinear form*

$$\langle \cdot, \cdot \rangle : S^p(V^*) \times S^p(V) \longrightarrow \mathbb{C},$$

by setting $\langle u^I, x^J \rangle = I! \cdot \delta_{I,J}$. There is a natural action of $GL(V)$ on $S^p(V^*) \times S^p(V)$ induced by the natural action of $GL(V)$ on $S^p(V)$ and the contragredient action of $GL(V)$ on $S^p(V^*)$. The apolar form is nondegenerate and invariant under this action, i.e., $\langle A \cdot g, A \cdot f \rangle = \langle g, f \rangle$ where $A \in GL(V)$, $f \in S^p(V)$, and $g \in S^p(V^*)$.

Definition 3.1.1. Let $f \in S^r(V)$ and $g \in S^p(V^*)$. We say that f is *apolar* to g if at least one of the following two conditions hold:

- (1) $r \leq p$ and $\langle g, f \cdot h \rangle = 0$ for all $h \in S^{p-r}(V)$;
- (2) $r \geq p$ and $\langle g \cdot h, f \rangle = 0$ for all $h \in S^{r-p}(V^*)$.

3.2. The apolarity theorem. Fix $p \geq 0$. Let d_1, \dots, d_r be nonnegative integers and let Ω be a finite subset of \mathbb{N}^r . For each r -tuple $I \in \Omega$, let t_I be a homogeneous symmetric tensor over V . Assume that for all $I \in \Omega$ we have

$$i_1 d_1 + i_2 d_2 + \dots + i_r d_r + \deg(t_I) = p.$$

This data determines a proposed canonical form; that is, we may propose that a generic element of $S^p(V)$ can be written as

$$\sum_{I \in \Omega} t_I \cdot s_1^{i_1} \cdots s_r^{i_r},$$

where each $s_j \in S^{d_j}(V)$.

The following theorem characterizes those proposed canonical forms which are, in fact, canonical. Its origins can be traced back to work of E. Lasker [2] and E. K. Wakeford [3]. The present formulation, as well as its proof, can be found in [1].

Theorem 3.2.1. *A generic element of $S^p(V)$ can be written as*

$$F = \sum_{I \in \Omega} t_I \cdot s_1^{i_1} \cdots s_r^{i_r}$$

for some $s_j \in S^{d_j}(V)$ if and only if there exist $s'_j \in S^{d_j}(V)$ such that the only $t \in S^p(V^*)$ apolar to all the $\left(\frac{\partial F}{\partial s_j}\right)_{s_j=s'_j}$ is zero.

4. REMOVABILITY OF MONOMIALS

Let $T(q, p)$ denote the set of all q -tuples $I = (i_1, \dots, i_q)$ of nonnegative integers satisfying $i_1 + \dots + i_q = p$. We say that a subset B of $T(q, p)$ is *removable* if a generic element of $S^p(V)$ can be written as

$$\sum_{I \in T(q, p) \setminus B} c_I \cdot X_1^{i_1} \cdots X_q^{i_q},$$

for some $c_I \in \mathbb{C}$ and $X_1, \dots, X_q \in V$. In this section, we attempt to describe as completely as possible those subsets B of $T(q, p)$ which are removable. Assume that $p, q > 1$ throughout.

4.1. The size of removable sets.

Proposition 4.1.1. *If $B \subseteq T(q, p)$ is removable, then $|B| \leq q(q-1)$.*

Proof. Let $B \subseteq T(q, p)$ satisfy $|B| > q(q-1)$. We show that

$$F(c_I; X_r) = \sum_{I \in T(q, p) \setminus B} c_I \cdot X_1^{i_1} \cdots X_q^{i_q}$$

is noncanonical. Let $c'_I \in \mathbb{C}$ and $X'_r \in V$.

If the X'_r are linearly dependent, we may assume, making a linear change of variables if necessary, that each X'_r lies in the span of x_1, \dots, x_{q-1} . Then $g = u_q^p$ is apolar to $\frac{\partial F}{\partial c_I}(c'_I; X'_r)$, $\frac{\partial F}{\partial X_r}(c'_I; X'_r)$ for all $I \in T(q, p) \setminus B$ and $r = 1, \dots, q$.

If instead the X'_r are linearly independent, we may assume, again making a linear change of variables if necessary, that each $X'_r = x_r$. Any $g \in S^p(V^*)$ apolar to the $\frac{\partial F}{\partial c_I}(c'_I; x_r)$ for $I \in T(q, p) \setminus B$ must have the form

$$g = \sum_{I \in B} a_I \cdot u^I.$$

We aim to show that there exists a nonzero such g apolar to the

$$\frac{\partial F}{\partial X_r}(c'_I; x_r) = \sum_{I \in T(q, p) \setminus B} i_r \cdot c'_I \cdot \mathbf{x}^{I-e_r}$$

for all r . Observe that g is apolar to $\frac{\partial F}{\partial X_r}(c'_I; x_r)$ for all r if and only if g is apolar to $\frac{\partial F}{\partial X_r}(c'_I; x_r) \cdot x_s$ for $r \neq s$. A non-trivial such g exists because these apolarity conditions produce $q(q-1)$ homogeneous linear equations in $|B| > q(q-1)$ unknowns a_I . By invariance of the apolar form and theorem 3.2.1, we have that F is noncanonical. Hence B is nonremovable, as desired.

4.2. The weighted biadjacency matrix. We turn our attention to subsets B of $T(q, p)$ of size $q(q-1)$. To each such subset B we associate a weighted bipartite graph $G_B = (N(G_B), E(G_B), W_{G_B})$, as follows. Let $D = \{e_i - e_j \mid 1 \leq i, j \leq q, i \neq j\}$. The nodes of G_B are given by the bipartition $N(G_B) = (B, D)$, and there is an edge $e(I, J)$ joining $I \in B$ to $J \in D$ if and only if $I + J \in T(q, p) \setminus B$. To each $I \in T(q, p) \setminus B$ associate a symbol c_I , and, letting $H_B = \{c_I \mid I \in T(q, p) \setminus B\}$, form the polynomial ring $\mathbb{C}[H_B]$. The weight map $W_{G_B} : E(G_B) \rightarrow \mathbb{C}[H_B]$ is given by

$$W_{G_B}(e(I, J)) = (i_k + 1) \cdot I! \cdot c_{I+J},$$

where $J = e_k - e_m$. As with any weighted bipartite graph, we may define its weighted biadjacency matrix, $M(G_B)$. We index $M(G_B)$ with the set $B \times D$, so that

$$M(G_B)_{I, J} = \begin{cases} (i_k + 1) \cdot I! \cdot c_{I+J} & \text{if } e(I, J) \text{ is an edge of } G_B; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the determinant of $M(G_B)$ is well-defined up to sign.

Proposition 4.2.1. *Let $B \subseteq T(q, p)$ have size $q(q-1)$. Then B is removable if and only if the determinant of $M(G_B)$ is nonzero.*

Proof. We have the proposed canonical form

$$F(c_I; X_r) = \sum_{I \in T(q, p) \setminus B} c_I \cdot X_1^{i_1} \cdots X_q^{i_q}.$$

By theorem 3.2.1 and a change of variables argument, F is canonical if and only if we can set each $X_r = x_r$ and choose $c'_I \in \mathbb{C}$ so that the only $g \in S^p(V^*)$ apolar to all the $\frac{\partial F}{\partial c_I}(c'_I; x_r)$, $\frac{\partial F}{\partial X_r}(c'_I; x_r)$ is zero. So set each $X_r = x_r$, leaving the c_I to be determined. Any $g \in S^p(V^*)$ apolar to $\frac{\partial F}{\partial c_I}(c_I; x_r)$ for all $I \in T(q, p) \setminus B$ has the form

$$g = \sum_{I \in B} a_I \cdot u^I.$$

Further, g is apolar to $\frac{\partial F}{\partial X_r}(c_I; x_r)$ for all r if and only if g is apolar to $\frac{\partial F}{\partial X_r}(c_I; x_r) \cdot x_s$ for all $r \neq s$. These apolarity conditions produce a system of $q(q-1)$ homogeneous linear equations in the $q(q-1)$ unknowns a_I . The coefficient matrix of this system is the weighted biadjacency matrix $M(G_B)$, up to a permutation of rows and columns. The result follows.

4.3. Relation to matchings. Assume that G_B admits a perfect matching in the graph-theoretic sense. Such a perfect matching can be identified with a bijection $f : B \rightarrow D$ such that $I \in B$ is joined to $f(I) \in D$ for all $I \in B$. Henceforth, we shall always make this identification between perfect matchings in G_B and such bijections f . By analogy with definition 2.3.1, we define $m_f : T(q, p) \rightarrow \mathbb{Z}$ by letting $m_f(I) = \#\{J \in B \mid I = J + f(J)\}$. Observe that m_f vanishes on B .

Definition 4.3.1. We say that a perfect matching f of G_B is *finite-acyclic* if for any perfect matching g with $m_f = m_g$, we have $f = g$.

We use the term *finite-acyclic* here because we are working in the finite set $T(q, p)$ as opposed to \mathbb{Z}^q .

Remark 4.3.2. Not every G_B admits a perfect matching. For example, take $B = \{(p, 0), (p-1, 1)\} \subseteq T(2, p)$. Even if G_B admits a perfect matching, it may not admit a finite-acyclic matching. For instance, one can show that if

$$B = \{(3, 0, 0), (2, 1, 0), (1, 1, 1), (1, 0, 2), (0, 3, 0), (0, 0, 3)\} \subseteq T(3, 3),$$

there exists a perfect matching, but no finite-acyclic matching.

Proposition 4.3.3. *Let $B \subseteq T(q, p)$ have size $q(q-1)$. If G_B admits a finite-acyclic matching, then B is removable.*

Proof. Let $f : B \longrightarrow D$ be a finite-acyclic matching. The determinant of $M(G_B)$ is, up to sign,

$$\sum_{\sigma} \left((-1)^{\sigma} \cdot \prod_{I \in B} M(G_B)_{I, \sigma f(I)} \right),$$

where σ ranges over all permutations of D . There is a one-to-one correspondence between perfect matchings of G_B and the nonzero summands in the above expansion. The summand corresponding to $\sigma = 1$ is

$$\prod_{I \in B} M(G_B)_{I, f(I)} = \prod_{I \in B} W_{G_B}(e(I, f(I))) = K \cdot \prod_{I \in B} c_{I+f(I)} = K \cdot \prod_{J \in T(q,p) \setminus B} c_J^{m_f(J)},$$

where K is some positive constant. Since f is a finite-acyclic matching, this term is not cancelled in the above expansion. Hence the determinant of $M(G_B)$ is nonzero. By proposition 4.2.1, B is removable.

4.4. Interior q -tuples and removability.

Definition 4.4.1. Let $I \in T(q, p)$. We say that I is an *interior q -tuple* if $I + J \in T(q, p)$ for all $J \in D$.

We are ready to state the main theorem of this section. The following theorem answers, to a large extent, a question raised by Wakeford in [3].

Theorem 4.4.2. *Let $B \subseteq T(q, p)$ consist of $q(q-1)$ interior q -tuples. Then B is removable.*

Proof. Let $\hat{T}(q, p) = \{(i_1, \dots, i_q) \in \mathbb{Z}^q \mid i_1 + \dots + i_q = p\}$. Fix any point $P \in \hat{T}(q, p)$. Under the injective map $\phi : \hat{T}(q, p) \hookrightarrow \mathbb{Z}^q$ given by $I \mapsto I - P$, we see that $\hat{T}(q, p)$ can be given the structure of a \mathbb{Z}^{q-1} -lattice with lattice basis given by $\Delta = \{e_i - e_{i+1} \mid 1 \leq i < q\} \subseteq D$. The dimension of the lattice is $q-1$ because $\text{Im}(\phi) = \{(i_1, \dots, i_q) \in \mathbb{Z}^q \mid i_1 + \dots + i_q = 0\}$. The set D is isomorphic as a root system to the roots of the Lie algebra of type A_{q-1} .

To see this, let $(\cdot, \cdot) : \mathbb{Z}^{q-1} \times \mathbb{Z}^{q-1} \longrightarrow \mathbb{Z}$ be defined by the restriction of the standard form $(e_i, e_j) = \delta_{ij}$ on \mathbb{Z}^q . The elements of Δ form a Dynkin diagram of type A_{q-1} , and D is precisely the set of elements in \mathbb{Z}^{q-1} of length $\sqrt{2}$.

Since $T(q, p)$ embeds naturally in $\hat{T}(q, p)$, which is in bijective correspondence to \mathbb{Z}^{q-1} , we see that B may be regarded as embedded in \mathbb{Z}^{q-1} .

By theorem 2.3.6, there exists an acyclic matching $\hat{f} : B \longrightarrow D$ when B is regarded as a subset of \mathbb{Z}^{q-1} under the various indentifications above. Using the fact that B consists of interior q -tuples we see that \hat{f} gives rise to a finite-acyclic matching $f : B \longrightarrow D$.

By proposition 4.3.3, we see that B is removable and the theorem follows.

Remark 4.4.3. Note that it also follows that any $d(d+1)$ monomial terms from the interior of a d -dimensional face of $T(p, q)$ are removable.

5. CONCLUSION

We hope that the techniques presented here have more general applicability, especially in the direction of generalizing these statements to arbitrary Lie groups. For instance, one can show the basic fact that semi-simple elements in a simple Lie algebra are generic by applying a slightly more general version of proposition 4.3.3. Specifically, if one replaces $T(q, p)$ with the weights of the Lie algebra and replaces B and D with the set of roots, one finds that $f : B \rightarrow D, \alpha \mapsto -\alpha$ is acyclic. Using techniques implicit in this paper, one concludes that the generic element of a simple Lie algebra is conjugate to an element in the Cartan, which consists of semi-simple elements.

Finally, we hope that the hypothesis on D in theorem 2.3.6 is unnecessary. Note that this hypothesis is satisfied by the simply laced root systems A , D , and E , but not the non-simply laced root systems B , C , F , and G .

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